On the Langevin Description of Non-Equilibrium Quantum Fields Dynamics

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Abstract

We examine the conditions under which the full nonequilibrium quantum dynamics of a scalar field can be modelled by a Markovian Langevin equation.

Introduction

- The dynamics of out-of-equilibrium quantum fields plays a key role in many topical areas of physics as
  - ◇ ion collisions,
  - ◇ early universe cosmology,
  - ◇ ultra cold atomic gases.
- Major breakthroughs have been achieved in recent years, in particular with the use of two-particle-irreducible functional techniques which allow for reliable calculations of the far-from-equilibrium dynamics.
- However there are physical situations where these techniques are difficult to implement.One way of coping with this is to employ adequate effective descriptions such as kinetic equations.
- Effective Langevin equations, from basic quantum field theory have been much studied since the pioneering work of Feynman and Vernon, based on the so-called influence functional formalism, where one typically obtains non-local – i.e. non-Markovian – Langevin-like equations with various additive and/or multiplicative noises.
- Assuming a clear separation of scales between the "memory time", and the "relaxation time", which characterize the dynamics of the relevant degrees of freedom, allows one to do further drastic simplifications, namely the assumption of effective local damping and Markovian noise.
- Existing calculations in simple theories show that memory kernels and noise correlators typically decay as power law in time, thereby questioning the existence of a memory time and more generally the possibility of a local, Markovian limit.
- We address this issue in this poster. We show that for typical situations there exist a range of time – not too short but not too long – where the actual solution can indeed be viewed as Markovian despite the memory kernels.
- This provides a QFT analysis of known results in quantum mechanics of non equilibrium relaxation.

I General settings

- We consider the non equilibrium dynamics of a scalar field. For Gaussian initial conditions this require the use of a set of two independent two-point functions

$$\begin{aligned}\rho(x,y) &= i\langle[\Phi(x),\Phi(y)]\rangle\\ F(x,y) &= \frac{1}{2}\langle\{\Phi(x),\Phi(y)\}\rangle\end{aligned}$$

- The spectral and statistical functions obey the non-equilibrium Schwinger-Dyson coupled equations

$$\begin{aligned}\left[\Box_x + M^2(x)\right]F(x,y) &= -\int_0^{x^0}\mathrm{d}^4z\,\Sigma_\rho(x,z)F(z,y)\\ &\quad+\int_0^{y^0}\mathrm{d}^4z\,\Sigma_F(x,z)\rho(z,y) \qquad (1)\\ \left[\Box_x + M^2(x)\right]\rho(x,y) &= -\int_{y^0}^{x^0}\mathrm{d}^4z\,\Sigma_\rho(x,z)\rho(z,y) \qquad (2)\end{aligned}$$

- These are causal equations which exhibit typical memory integrals dealing with the history of the evolution.

II Langevin process

- F can be seen as the two-point correlation function of oscillators driven by noise related to the statistical part of the self energy, if one considers a collection of oscillators obeying the equation :

$$\left[\Box_x + M^2(x)\right]\varphi^\xi(x) + \int_0^{x^0}\mathrm{d}^4z\,\Sigma_\rho(x,z)\varphi^\xi(z) = \xi(x)$$

- and one sets :

$$\begin{aligned}\Sigma_F(x,y) &\equiv -\overline{\xi(x)\xi(y)}\\ F(x,y) &\equiv -\overline{\langle\varphi^\xi(x)\varphi^\xi(y)\rangle} \qquad (3)\end{aligned}$$

where $\overline{\dots}$ is the average over noise realization , $\langle\langle\dots\rangle\rangle$ is the average over initial conditions. The quantum one is then $\langle\dots\rangle = \overline{\langle\langle\dots\rangle\rangle}$

- One recovers exactly the Schwinger-Dyson equation for F and ρ

III Approximation Scheme

- Here we make the assumption that the self energies decay « fast enough », such that under the integrals a harmonic approximation can be performed :

$$F_p(u,t') \approx F_p(t,t')\cos\epsilon_p(t)(u-t) - \frac{\partial_t F_p(t,t')}{\epsilon_p(t)}\sin\epsilon_p(t)(u-t) \quad (4)$$

Which gives under the integrals

$$\begin{aligned}\int_{-\infty}^{t}\mathrm{d}u\,\Sigma_p^\rho(t,u)F_p(u,t') &= \int_{-\infty}^{+\infty}\mathrm{d}u\,\Sigma_p^R(t,u)F_p(u,t')\\ &\approx \delta\epsilon_p^2(t)F_p(t,t') + 2\bar\gamma_p(t)\partial_t F_p(t,t')\end{aligned}$$

Where

$$\begin{aligned}\delta\epsilon_p^2(t) &\equiv \mathrm{Re}\Sigma_p^R(t;\epsilon_p(t))\\ \bar\gamma_p(t) &\equiv -\frac{\mathrm{Im}\Sigma_p^R(t;\epsilon_p(t))}{2\epsilon_p(t)} \qquad (5)\end{aligned}$$

and the retarded self energy is given by

$$\Sigma_p^R(t,t') = \theta(t-t')\Sigma_p^\rho(t,t') = \int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\Sigma_p^R(t;\omega)$$

- Such that the equation of motion reads

$$\begin{aligned}&(\partial_t^2 + 2\bar\gamma_p(t)\partial_t + M_p^2(t) + \delta\epsilon_p^2(t))F_p(t,t') =\\ &\int_{-\infty}^{+\infty}\mathrm{d}u\;\Sigma_{F,p}(t,u)G_p^A(u,t') \qquad (6)\end{aligned}$$

With

$$\left[\partial_t^2 + 2\bar\gamma_p(t)\partial_t + \bar\epsilon_p^2(t)\right]G_p^{R,A}(t,t') = \delta(t-t')$$

- This approximation allows one to get a local damping like term in the equation, without any assumptions on the noise kernel.

IV Exactly solvable example

- To understand the domain of validity of the approximation made above we focus on a near stationary situation where $\Sigma_p^{F,\rho}(t,t') \equiv \Sigma_p^{F,\rho}(t-t')$, such that the spectral function is also time translation invariant $\rho(t,t') = \rho(t-t')$
- The solution is then given by

$$F_p(t,t') = F_p^{\mathrm{eq}}(t-t') + \theta(t)\theta(t')F_p^K(t,t') \qquad (7)$$

Where the non equilibrium part is given by

$$\begin{aligned}F_p^K(t,t') &= A_p\rho_p(t)\rho_p(t') + B_p\dot\rho_p(t)\dot\rho_p(t') \qquad (8)\\ &\quad + C_p\left[\dot\rho_p(t)\rho_p(t') + \rho_p(t)\dot\rho_p(t')\right]\end{aligned}$$

- For the equilibrium contribution we decompose it into a Breit-Wigner part and a rest

$$\begin{aligned}\rho_p(t) &= Z_p\frac{\sin\epsilon_p t}{\epsilon_p}e^{-\gamma_p|t|} + \rho_p^{\mathrm{pow}}(t) \qquad (9)\\ F_p^{\mathrm{eq}}(t) &= \kappa_p\left(\cos\epsilon_p t + \frac{\gamma_p}{\epsilon_p}\sin\epsilon_p|t|\right)e^{-\gamma_p|t|} + F_p^{\mathrm{pow}}(t) \qquad (10)\end{aligned}$$

Where

$$\begin{aligned}Z_p^{-1} &= 1 - \frac{d}{d\omega^2}\mathrm{Re}\tilde\Sigma_p^R(\omega)\Big|_{\omega=\epsilon_p}\\ \kappa_p &= \frac{Z_p}{\epsilon_p}\left(n\left(\beta\epsilon_p\right) + \frac{1}{2}\right)\end{aligned}$$

- The non Breit-Wigner terms are typically power laws, at long time, dictated by cuts and thresholds.
- For time short enough $e^{-\gamma_p|t|}/\epsilon_p \gg \rho_{pow} \propto (\mu t)^{-\nu}$. The characteristic time scale is

$$\gamma t^L \approx \ln\left(\frac{\mu}{\gamma}\right)^\nu \qquad (11)$$

- Looking at Eq. (8), (9) and (10) one recognizes the equilibrium solutiona damped oscillator driven by a Markovian , and the approach towards it, with

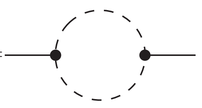
$$\left[\partial_t^2 + 2\gamma_p\partial_t + \epsilon_p^2\right]\varphi_p^\xi(t) = \xi_p(t) \qquad (12)$$

and,

$$\overline{\xi_p(t)\xi_p(t)} \equiv -4\gamma_p\epsilon_p(n(\epsilon_p)+1/2)\delta(t)$$

V Explicit Example

- For illustration purpose, we consider a $\varphi\chi^2$ model. The one loop φ-self energy is

$\Sigma_\varphi =$ 

- Assuming χ in thermal equilibrium and no backreaction, the late time behavior is dictated by the two-particle threshold and reads

$$\begin{aligned}\Sigma_\varphi^\rho(t) &\approx \sigma_\rho^{\mathrm{pow}}(t) = -g^2a_\rho\frac{\cos\left(2mt+\pi/4\right)}{(mt)^{3/2}} \qquad (13)\\ \Sigma_\varphi^F(t) &\approx \sigma_F^{\mathrm{pow}}(t) = g^2a_F\frac{\sin\left(2mt+\pi/4\right)}{(mt)^{3/2}} + \kappa \qquad (14)\end{aligned}$$

- Computing the propagators gives

$$\begin{aligned}\rho_\varphi(t) &\approx \frac{\sin\left(Mt\right)}{M}e^{-\gamma|t|} + \rho_\varphi^{\mathrm{pow}}(t)\\ F_\varphi^{\mathrm{eq}}(t) &\approx \left(n\left(\beta M\right)+\frac{1}{2}\right)\frac{\cos Mt}{M}e^{-\gamma|t|} + F_\varphi^{\mathrm{pow}}(t)\end{aligned}$$

Where

$$\rho_\varphi^{\mathrm{pow}}(t) = -\left|\tilde G_\varphi^R(2m)\right|^2\sigma_\rho^{\mathrm{pow}}(t) \approx \frac{\sigma_\rho^{\mathrm{pow}}(t)}{(M^2-4m^2)^2}$$

and

$$\begin{aligned}F_\varphi^{\mathrm{pow}}(t) &= -\left|\tilde G_\varphi^R(2m)\right|^2\sigma_F^{\mathrm{pow}}(t) + \kappa\left|\tilde G_\varphi^R(0)\right|^2\\ &\approx \frac{\sigma_F^{\mathrm{pow}}(t)}{(M^2-4m^2)^2} + \frac{\kappa}{M^4}\end{aligned}$$

- Such that for time smaller than the Langevin time given below, the dynamics behaves essentially as a Markovian process

$$\gamma t^L \approx \ln\left\{\left(\frac{m}{\gamma}\right)^{3/2}\frac{M}{m}\left(1-\frac{4m^2}{M^2}\right)^{5/2}\frac{n\left(\frac{\beta M}{2}\right)+\frac{1}{2}}{n(\beta m)+\frac{1}{2}}\right\} \qquad (15)$$

- This time range is enough for an almost complete decay and gives the dominant contribution in the narrow width case.

VI Consistency Check

- It is instructive to understand how the above approximation (neglect of long time power laws) can be obtain directly at the level of the equations of motion, as explained above. For this purpose one can plug back the solution of the effective equation in the general one

$$\begin{aligned}&\int_0^t d\tau\,\Sigma_p^\rho(t-\tau)\rho_p^{\mathrm{eff}}(\tau) =\\ &= \rho_p^{\mathrm{eff}}(t)\mathrm{Re}\,\sigma_p^{i\epsilon_p+\gamma_p}(t) - \frac{\dot\rho_p^{\mathrm{eff}}(t)+\gamma_p\rho_p^{\mathrm{eff}}(t)}{\epsilon_p}\mathrm{Im}\,\sigma_p^{i\epsilon_p+\gamma_p}(t)\end{aligned} \qquad (16)$$

where,

$$\sigma_p^z(t) = \int_0^t d\tau\,\Sigma_p^\rho(\tau)e^{z\tau} = \int_{-\infty}^t d\tau\,\Sigma_p^R(\tau)e^{z\tau}$$

- A detailed analysis reveals

$$\mathrm{Re}\,\sigma_p^{i\epsilon_p+\gamma_p}(t) = \mathrm{Re}\,\tilde\Sigma_p^R(\epsilon_p)\left[1+\mathcal{O}\left(\frac{a_p^\rho}{\epsilon_p\delta\epsilon_p^2}\frac{e^{\gamma_p t}}{(\mu_p t)^\nu}\right)\right] \qquad (17)$$

$$\mathrm{Im}\,\sigma_p^{i\epsilon_p+\gamma_p}(t) = \mathrm{Im}\,\tilde\Sigma_p^R(\epsilon_p)\left[1+\mathcal{O}\left(\frac{a_p^\rho}{\gamma_p\epsilon_p^2}\frac{e^{\gamma_p t}}{(\mu_p t)^\nu}\right)\right] \qquad (18)$$

- One can easily check that for time smaller than Langevin time, the exponential corrections are negligible. This analysis is of particular importance for cases where one cannot compare to the exact solution and has to make approximations at the level of the e.o.m

Conclusion

- This work aimed to shed some light on the dynamics of out-of-equilibrium quantum scalar fields. It tried to understand when a local Langevin approximation can be motivated.
- After recalling the general setup where the quantum equation of motion can one to one related to a Langevin process, It set an approximation scheme which localizes the damping and noise kernels.
- But in an explicit example it exhibited the non local behavior of the kernels, due to cuts and multi particle thresholds, questioning the validity of the local approximation
- Despite the non local kernels and thus no true separation of scale, a local approximation is still valid in a certain time range, within which the dynamics is essentially markovian.