Nature of the thermal phase transition of the scalar ϕ^4 model in the two-loop approximation of the 2PI formalism

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ABSTRACT

We investigate numerically the temperature phase transition of the Euclidean ϕ^4 theory using the 2PI effective potential truncated at 2-loop order. Fast Fourier Transform algorithms and accelerated Matsubara sums are combined to achieve a well controlled, accurate solution of the fully renormalized coupled set of propagator and field equations. We find that if for a set of mass and coupling parameters a phase transition occurs, then it is of second order. Analytical and numerical determination of the static critical exponents give mean-field values. However, if we implement some ideas of the renormalization group, by letting the parameters run with the temperature, the values depart from mean-field.

MOTIVATION

It is known that the ϕ^4 model has a second order phase transition which is not reproduced by conventional perturbation theory. Therefore we use a systematically improvable resummation scheme, the 2PI formalism. The first steps were taken in [1] where the Hartree approximation was considered. In the region of parameters where a phase transition was found it was analytically proved to be of first order. In [2] the 2-loop approximation was addressed in Minkowski space. The order of the phase transition turned out to be second order, but no critical exponents were determined. Since a good description of a phase transition should reproduce the correct exponents, our goal was to confirm, improve the accuracy and extend the results of [2].

EQUATIONS & RENORMALIZATION

Equations

The action of the Euclidean ϕ^4 theory:

$$S[\varphi] \equiv \int_0^{1/T} d\tau \int d^3x \left(\frac{1}{2}(\partial_\tau \varphi)^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{m_0^2}{2}\varphi^2 + \frac{\lambda_0}{4!}\varphi^4 \right).$$

The 2PI effective potential, truncated at the second loop order:

$$\gamma[\phi,G] \equiv \frac{1}{2}\int_Q^T \left[\ln G^{-1}(Q) + (Q^2+m_0^2)G(Q) - 1\right] + \frac{m_2^2}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + \frac{\lambda_2}{4}\phi^2 \;\bigcirc\; + \frac{\lambda_0}{8}\;\infty\; - \frac{\lambda_\star^2}{12}\phi^2 \;\ominus\;.$$

Propagator equation at any ϕ, determining $\bar{G}(K) = 1/(K^2+\bar{M}^2(K))$:

$$0 = \left.\frac{\delta\gamma[\phi,G]}{\delta G}\right|_{\bar{G}} = m_0^2 - \bar{M}^2(K) + \frac{\lambda_2}{2}\phi^2 + \frac{\lambda_0}{2}\;\bigcirc\; - \frac{\lambda_\star^2}{2}\phi^2 \;\overset{K}{\bigcirc}\;\overset{K}{}.$$

Field equation at the solution for the propagator:

$$0 = \left.\frac{\delta\gamma[\phi,G]}{\delta\phi}\right|_{\bar{G}_{\bar{\phi}},\bar{\phi}} = \bar{\phi}\left(m_2^2 + \frac{\lambda_4}{6}\bar{\phi}^2 + \frac{\lambda_2}{2}\;\bigcirc\; - \frac{\lambda_\star^2}{6}\;\ominus\;\right).$$

Curvature (second derivative) of the potential:

$$\left.\frac{\delta^2\gamma[\phi,G]}{\delta\phi^2}\right|_{\bar{G}_0,\phi=0} = \hat{M}^2_{\phi=0} = m_2^2 + \frac{\lambda_2}{2}\;\bigcirc\; - \frac{\lambda_\star^2}{6}\;\ominus\;.$$

Notation: $\int_Q^T \equiv T\sum_{\omega_n}\int \frac{d^3q}{(2\pi)^3}\Theta(\Lambda-|q|)$, with $\omega_n = 2\pi T n$ the Matsubara frequency of a boson and Λ the 3-dimensional cutoff.

Renormalization

As a truncation artefact two- and four-point functions have different definitions even in the symmetric phase (Note: this is why we need m_0^2 and m_2^2 as well as λ_0, λ_2 and λ_4 separately):

$$\hat{M}^2_{\phi=0},\;\; \bar{M}^2_{\phi=0}$$

$$\hat{V}_{\phi=0} \equiv \left.\frac{\delta^4\gamma[\phi,G]}{\delta\phi^4}\right|_{\bar{G}_0,\phi=0},\;\; V_{\phi=0} \equiv \left.\frac{\partial^2\bar{M}^2(K)}{\partial\phi^2}\right|_{\phi=0,K=0},$$

$$\bar{V}_{\phi=0} \equiv \left[\frac{1}{\lambda_0} + \frac{1}{2}\;\overset{K}{\bigcirc}\;\overset{K}{}\right]^{-1}_{\phi=0,K=0}.$$

In the full theory the different definitions coincide. We encode this behaviour in our renormalization coinditions at large enough temperature $T_\star$, where the system is required to be in its symmetric phase. We prescribe two renormalization and three consistency conditions[1]:

$$\hat{M}^2_{\phi=0,T_\star} = \bar{M}^2_{\phi=0,T_\star} = m_\star^2 \;;\; \hat{V}^{T_\star}_{\phi=0} = V^{T_\star}_{\phi=0} = \bar{V}^{T_\star}_{\phi=0} = \lambda_\star.$$

Although there are more conditions, only two "measured" parameters are required.
The renormalized effective potential is defined by

$$\Delta\gamma[\phi,\bar{G}_\phi]^T = \gamma[\phi,\bar{G}_\phi]^T - \gamma[0,\bar{G}_0]^{T_\star}.$$

NUMERICS

There are two types of sum-integrals to compute. The bubble diagram in the propagator equation is a convolution, while the tadpole or setting-sun diagrams are momentum independent. All functions are rotation invariant and we use an $N_\tau \times N_s$ grid to store the propagator (N_τ non-negative Matsubara frequencies and N_s values of the modulus of the momentum). We use Fast Fourier Transform algorithms[3] for the convolutions both for the Matsubara frequencies and the momentum:

$$\mathcal{C}_{N_\tau,N_s}[f,g](K) = \left[\int_Q^T f(Q)\,g(K-Q)\right]_{N_\tau,N_s} = \mathcal{F}_{N_\tau,N_s}\left[\mathcal{F}^{-1}_{N_\tau,N_s}[f]\mathcal{F}^{-1}_{N_\tau,N_s}[g]\right](K).$$

In the other case we sum up the Matsubara frequencies and use a trapezoid rule[4] for the momentum integrals:

$$\mathcal{V}_{N_\tau,N_s}[f] = \left[\int_Q^T f(Q)\right]_{N_\tau,N_s} = T\left[\int\frac{d^3q}{(2\pi)^3}\right]_{N_s} f(0,q) + 2T\sum_{n=1}^{N_\tau-1}\left[\int\frac{d^3q}{(2\pi)^3}\right]_{N_s} f(i\omega_n,q).$$

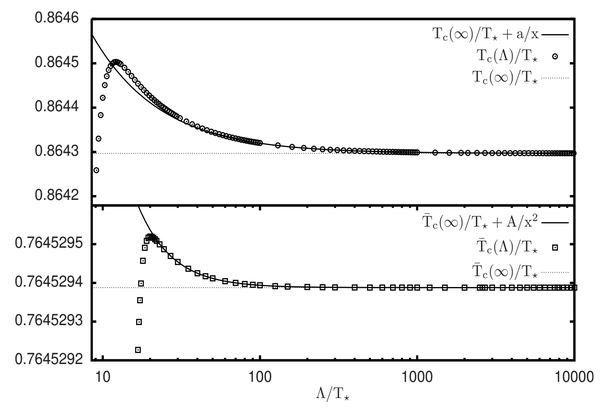
These are slowly converging methods as $N_\tau,\; N_s \to \infty$ for slowly decreasing functions, but we may reshuffle our equations using the explicit form of the counterterms such that we only need to compute such sum-integrals for differences of propagators which decrease faster in the UV. For more details see [5].

CRITICAL TEMPERATURE

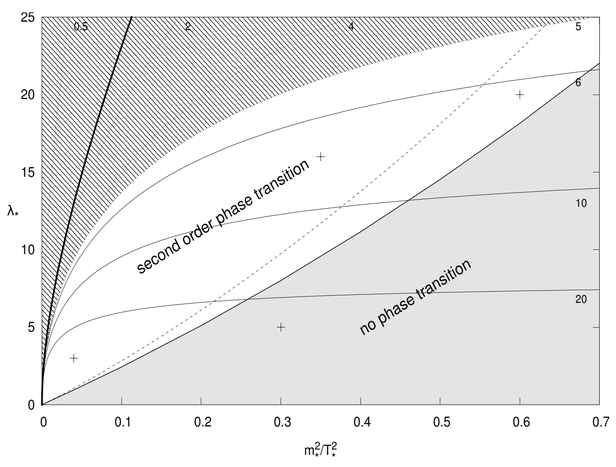
There are two possibilities for defining a critcial temperature:

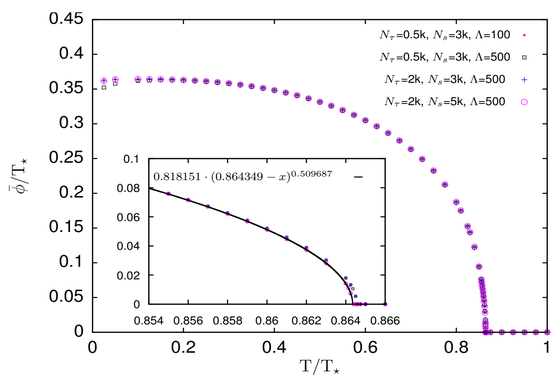
$$\hat{M}^2_{\phi=0}(T_c) = 0, \quad \bar{M}^2_{\phi=0}(\bar{T}_c) = 0.$$

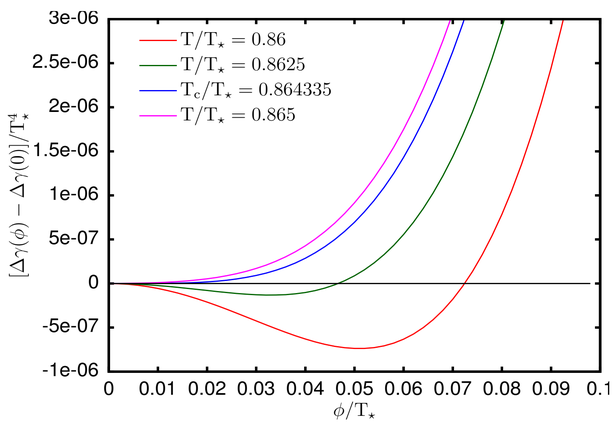
T_c and $\bar{T}_c$ can be determined almost exactly by performing the Matsubara sums analytically and using advanced integration routines for the remaining integral, up to arbitrary precision. Alternatively, we can use our full numerical machinery and check the errors originating from our method.
Both critical temperatures can be determined in cutoff or dimensional regularization. This way we can check that renormalization is properly done, since cutoff effects die out as $\Lambda \to \infty$.



PHASE TRANSITION







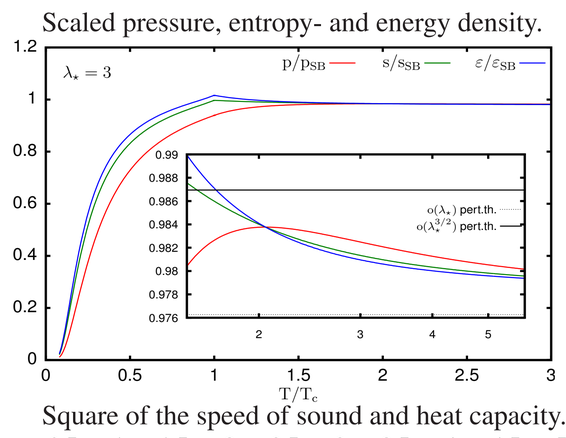
Top left: The solid line represents the parameter sets at which $T_c = 0$. The dashed line shows $\bar{T}_c = 0$, which becomes the critical temperature for the RG-improved case. The shaded region indicates where the Landau pole is not large enough, compared to all other scales, for the model to make sense.

Top right: The expectation value of the field as a function of the temperature for several sets of discretization parameters. The inset is zoom-in into the region around the critical temperature. Also a fit is plotted here which is consistent with the mean-field value of the critical exponent $\beta = 1/2$.

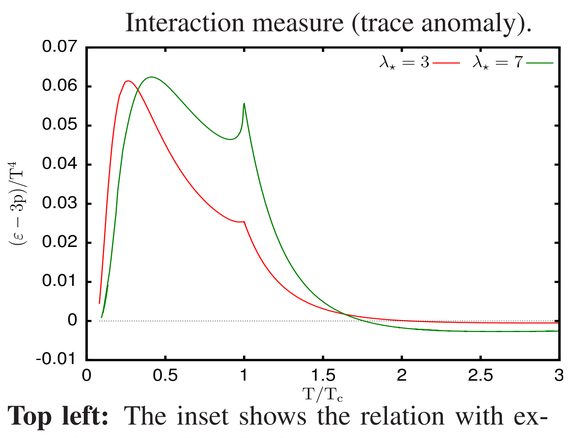
Bottom left: The potential for several temperatures around T_c. The abscence of a second minimum indicates also a second order phase transition.

THERMODYNAMICS

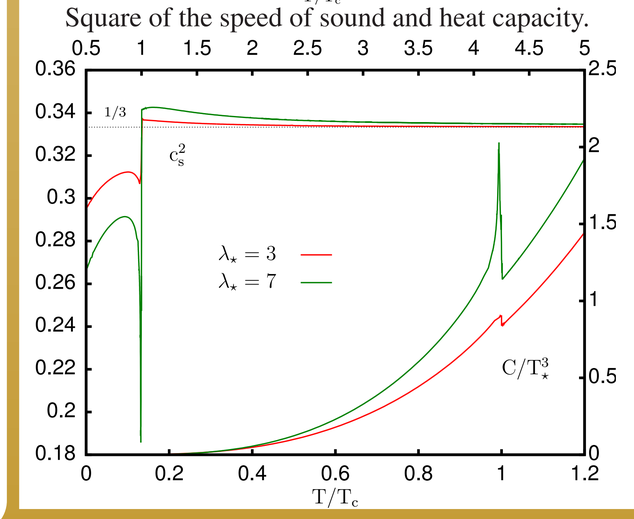
Scaled pressure, entropy- and energy density.



Interaction measure (trace anomaly).



Square of the speed of sound and heat capacity.



Top left: The inset shows the relation with expectations based on the first two orders perturbative expansion[6]. For larger coupling the curves go out of the region bounded by the perturbative results, which shows the weakness of perturbation theory.

Top right: The sharp cusp indicates the phase transition temperature. Both curves approach zero for high T from below.

Bottom left: The square of the speed of sound goes to the conformal limit $1/3$ at high T. The discontinuity of the heat capacity at T_c implies that the critical exponent α is zero, which is the mean-field value.

CRITICAL EXPONENTS, RG-IMPROVEMENT

In the discussed two-loop approximation all static critical exponents are of mean-field value as we have seen for α and β. To depart form the mean-field we employ a renormalization group inspired modification. We let $m_\star^2$ and $\lambda_\star$ run with the temperature in such a way that the bare parameters m_0^2 and λ_0 remain constant. From

$$\frac{dm_0^2}{dT_\star} = 0, \quad \frac{d\lambda_0}{dT_\star} = 0$$

we obtain differential equations for $m_\star^2(T_\star)$ and $\lambda_\star(T_\star)$. As initital conditions we use the original parameters at the original $T_\star$. The solutions are such that

$$m_\star^2(\bar{T}_c) = \lambda_\star(\bar{T}_c) = 0.$$

The running repairs the discrepancies mentioned in connection with the renormalization, having now:

$$\hat{M}^2_{\phi=0} = \bar{M}^2_{\phi=0}\;; \qquad V_{\phi=0} = \hat{V}_{\phi=0} = \bar{V}_{\phi=0}\;,$$

at any temperature. This also leads to $T_c = \bar{T}_c$. The main problem of the RG-improvement is that it is only defined in the symmetric phase as $m_\star^2$ would be negative for $T < \bar{T}_c$. This puts a restriction on the critical exponents which can be obtained.
Interestingly the self-energy becomes momentum independent for any ϕ at the critical temperature making it possible to calculate all accessible exponents analytically. The following changes are achieved:

$$\delta = 3 \to 5, \quad \gamma = 1 \to 2, \quad \nu = \frac{1}{2} \to 1.$$

CONCLUSIONS

- We solved the fully renormalized finite temperature ϕ^4 model in 2-loop approximation of the 2PI.
- We found that the phase transition is of 2nd order in the parameter region where it was 1st order in the Hartree apporximation. This also confirms the results in [2].
- We obtained that the critical exponents have mean-field values.
- A RG-improvement moves the critical exponents away from the mean-field values.
- We developed a well controlled, accurate, numerical method for solving the model.

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