Preliminary results from the 4PI effective action

in collaboration with E. Kovalchuk, Yun Guo, Weijie Fu

Outline:

- Introduction to nPI
- 4PI scalar theory - structure of renormalization
- Results in a toy (toy) model

2PI for Scalar Theories:

generating functional with local and bi-local sources

$$Z[J,K] = e^{iW[J,K]} = \int \mathcal{D}\hat{\phi} e^{i(S[\hat{\phi}] + J_i \hat{\phi}^i + \frac{1}{2} R_{ij} \hat{\phi}^i \hat{\phi}^j)}$$

short-hand notation:

$$\int dx \int dy \; \varphi(x) R(x,y) \varphi(y) \rightarrow \varphi_i R_{ij} \varphi_j \rightarrow R\varphi^2$$

Legendre transform:

$$\begin{aligned}\Gamma[\phi, G] &= W[J, K] - J_i \phi^i - \frac{1}{2}\phi^i R_{ij} \phi^j \\ &= S_{\rm cl}[\phi] + \frac{i}{2}{\rm Tr}\,{\rm ln} G^{-1} + \frac{i}{2}{\rm Tr}\, G_0^{-1}(G - G_0) + \Gamma_2[\phi, G]\end{aligned}$$

$\Gamma[\phi, G]$ is a functional of the 1- and 2-point functions

ϕ and G are determined self-consistently from the equations of motion

variational principle (in the absence of sources)

$$\frac{\delta\Gamma}{\delta\phi} = \frac{\delta\Gamma}{\delta G} = 0$$

$$\Rightarrow \quad G^{-1} = G_0^{-1} - \Sigma[\phi, G]\,, \qquad \Sigma[\phi, G] := 2i\frac{\delta\Gamma_2}{\delta G}$$

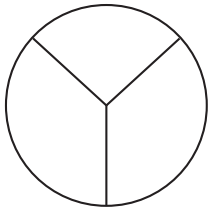
Compare to $\Gamma[\phi] =$ 1PI effective action:

- $\Gamma[\phi, G]$ depends on the self consistent propagator

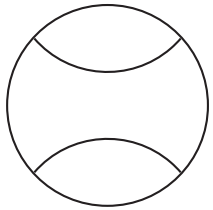
$\longrightarrow$ truncated $\Gamma[\phi, G]$ includes an infinite resummation of diagrams

$\longrightarrow$ non-perturbative

- $\Gamma[\phi, G]$ is 2PI - no double counting



$2PI$



$2PR$

nPI effective action

nPI Γ is a functional of n-point functions

3PI $\Gamma[\phi, G, U]$, 4PI $\Gamma[\phi, G, U, V] \cdots$

n-point functions determined self-consistently from the equations of motion

$\Rightarrow$ hierarchy of coupled equations - no exact solution method is available

$\Rightarrow$ use approximation techniques: truncate the effective action

goal: improve convergence relative to standard perturbation theory

compare: HTL effective theory (high Temperature gauge theories)

in principle nPI is valid arbitrarily far from equilibrium

Application: transport coefficients

calculation is complicated by the presence of singularities

very subtle power-counting arguments to see what contributes even at lo

integral equations that resum singularities are produced in a natural way

without any power-counting arguments

$\Rightarrow$ a natural framework to work beyond leading order

Results:

3PI: can reproduce lo kinetic theory results = direct connection to field theory

P. Arnold, G.D. Moore and L. G. Yaffe, JHEP **0305**, *051 (2003).*

MEC and E. Kovalchuk, Phys. Rev. D **76**, *045019 (2007) - σ qed.*

MEC and E. Kovalchuk, Phys. Rev. D **80**, *085013 (2009) - η qcd.*

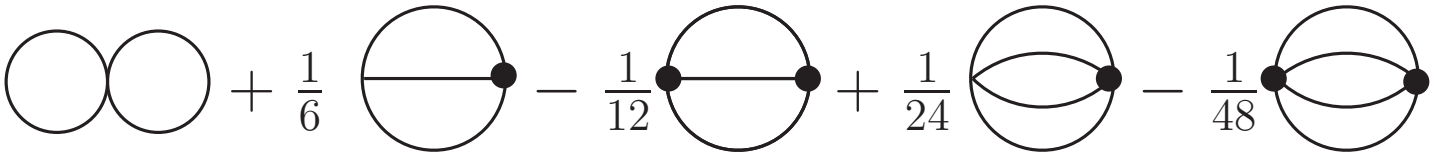
4PI: next-to-leading order transport coefficients

MEC and E. Kovalchuk, Phys. Rev. D **81**, *065017 (2010).*

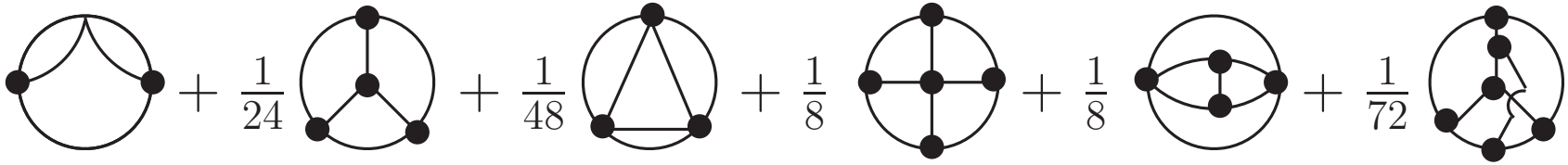
nlo - 4loop 4PI - scalar theory with cubic and quartic interaction

$$
\begin{aligned}
Z[J,R,R_3,R_4] &= \mathrm{Exp}[iW[J,R,R_3,R_4]] \\
&= \int \mathcal{D}\hat{\phi}\, \mathrm{Exp}[i(S_{\mathrm{cl}}[\hat{\phi}] + J\hat{\phi} + \frac{1}{2!}R\hat{\phi}^2 + \frac{1}{3!}R_3\hat{\phi}^3 + \frac{1}{4!}R_4\hat{\phi}^4)] \\
\Gamma[\phi,G,U,V] &= S_{\mathrm{cl}}[\phi] + \frac{i}{2}\mathrm{Tr}\,\mathrm{Ln}G_{12}^{-1} \\
&\quad + \frac{i}{2}\mathrm{Tr}\left[(G^0_{12}(\phi))^{-1}\left(G_{21} - G^0_{21}(\phi)\right)\right] - i\Phi[\phi,G,U,V]
\end{aligned}
$$

Φ =



EIGHT $\quad$ EGG_0 $\quad$ EGG $\quad$ BBALL_0 $\quad$ BBALL



HAIR $\quad$ MERCEDES $\quad$ LOOPY $\quad$ TARGET $\quad$ EYEBALL $\quad$ TWISTED

definitions of n-point functions

theory defined in terms of self-consistently determined n-point fcns

additional (different) definitions of vertices are possible

(equivalent if the action is not truncated)

will consider only the symmetric theory: $\tilde{\phi} = \tilde{U} = 0$

another 4-point vertex:

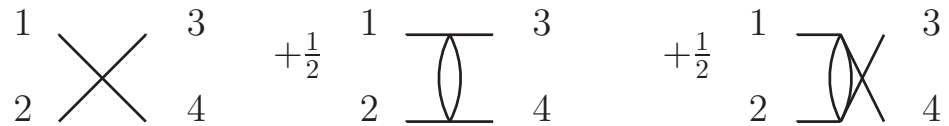
$$\Lambda = 4 \frac{\delta^2 \Phi}{\delta^2 G}\Big|_{\tilde{x}}$$

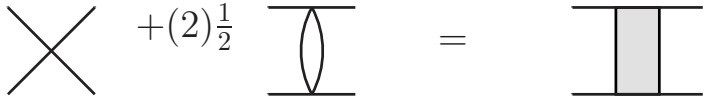
this vertex is 'diagonal' $\rightarrow$ $\Lambda(P, K)$

"Bethe-Salpeter" vertex:
$M = G^{-4} M^c$ - resumms Λ in the s-channel

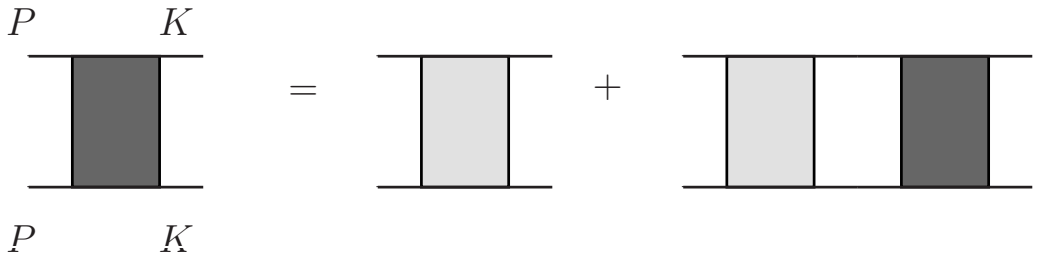
2PI example:

Φ = $\frac{1}{8}$  $+\frac{1}{48}$ 

$\Lambda_{020} = 4\frac{\delta^2\Phi}{G_{12}G_{34}}$ = 

= 

$M = \Lambda + \frac{1}{2}\Lambda G^2 M$



diagonal vertices impt - related to the 2-point function

Renormalization:

previous m, λ, G and V were bare quantities (unrenormalized)

bare quantities related to renormalized ones:

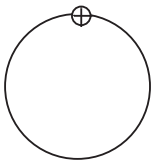
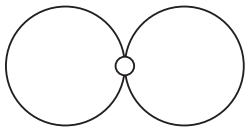
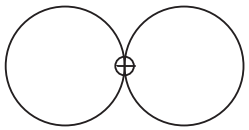
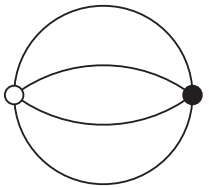
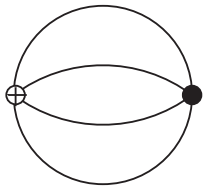
$$\delta m^2 = Z m_B^2 - m^2\,, \quad \delta\lambda = Z^2\lambda_B - \lambda\,, \quad \delta Z = Z - 1$$

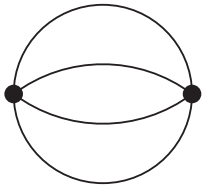
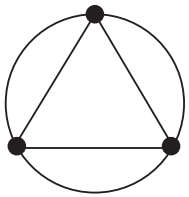
$$G_B = ZG\,, \quad V_B = Z^{-2}V\,,$$

$$ZG_{0B}^{-1} = G_0^{-1} + \delta G_0^{-1}\,, \quad \delta G_0^{-1} = i(\delta Z\Box + \delta m^2)$$

Γ in terms of renormalized quantities:

$$
\begin{aligned}
i\Gamma[G,V] = & \quad -\frac{1}{2}\mathrm{Tr}\ln G^{-1} - \frac{1}{2}\mathrm{Tr}G_0^{-1}G + \Phi_A[G,V] + \Phi_B[G,V] \\
\Phi_A = & \quad -\frac{1}{2}\mathrm{Tr}\delta G_0^{-1}G + \frac{1}{8}(\lambda+\delta\lambda_{\mathrm{et}})G^2 + \frac{1}{4!}(\lambda+\delta\lambda_{\mathrm{bb}})G^4V \\
\Phi_B = & \quad -\frac{1}{2}\frac{1}{4!}V^2G^4 + \frac{1}{48}V^3G^6
\end{aligned}
$$

$\Phi_{\mathrm{A}} =$ $-\frac{1}{2}$  $+\frac{1}{8}$  EIGHT $+\frac{1}{8}$  $\mathrm{EIGHT}_{\mathrm{ct}}$ $+\frac{1}{24}$  BBALL_0 $+\frac{1}{24}$  $\mathrm{BBALL}_{\mathrm{ct}}$

$\Phi_{\mathrm{B}} =$ $-\frac{1}{48}$  $+\frac{1}{48}$ 

Goal

show 4PI effective action has the right structure for renormalization

$\Rightarrow$ can renormalize V and $\Sigma\left[M\right]$ with $\delta\lambda_{\rm bb}$ and $\delta\lambda_{\rm et}$

- non-perturbative counter-terms

- momentum independent

- same to the order of the approximation

2PI renormalization:
J-P Blaizot, E. Iancu, U. Reinosa, Nucl. Phys. A736, 149 (2004).
H. Hees, J. Knoll, Phys. Rev. D65, 025010 (2002); D65 105005 (2002); D66 025028 (2002).
J. Berges, S. Borsanyi, U. Reinosa, J. Serreau, Annals Phys. 320, 344 (2005); JHEP 0607, 028, (2006).

NEED diagrams in V and M divided into:

- finite
- infinite constants
- infinite p-dependent terms

$\Big\}$ counter $-$ terms

Vertex Splitting

group factors in integrand $f(\{P_i\}, L)$ with $L \notin \{P_i\}$

choose the function f so that

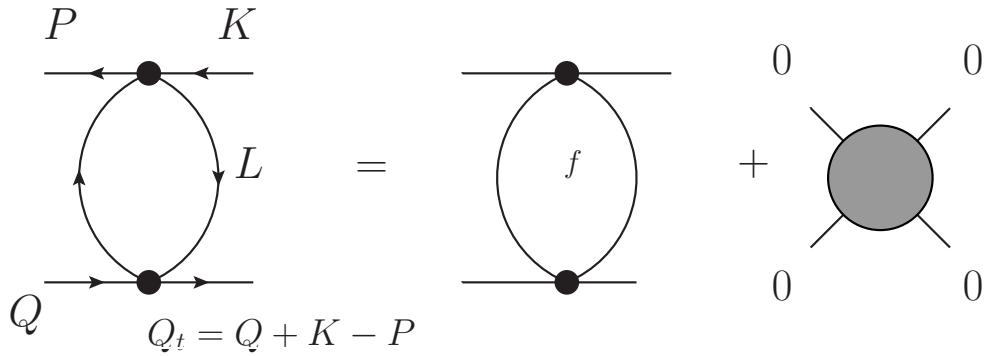
$$f(\{P_i\}, L) = \underbrace{[f(\{P_i\}, L) - f(0, L)]}_{\Delta_L} + f(0, L)\,, \quad \lim_{L\to\infty} \Delta_L \sim \frac{1}{L} f$$

- can show integrals we need at most logarithmically divergent

$\Rightarrow$ term with a factor $\Delta_L \sim \frac{1}{L} f$ is finite when integrated over L

$f(0, L)$ is independent of the momenta which are external to the loop

Example:



$$I_t^{VV}(-P,Q,K,-Q_t) = \int dL\, V(Q,-Q_t,L,-L_t)G_L G_{L+K-P} V(-L,L_t,-$$

define: $f(\{P_i\},L) = V(Q,-Q_t,L,-L_t)G_L G_{L+K-P} V(-L,L_t,-P,K)$

$$I_t^{VV}(-P,Q,K,-Q_t) = \int dL[\Delta_L + f(0,L)] = \text{finite} + I_1(V_0,V_0)$$

term with Δ_L is finite

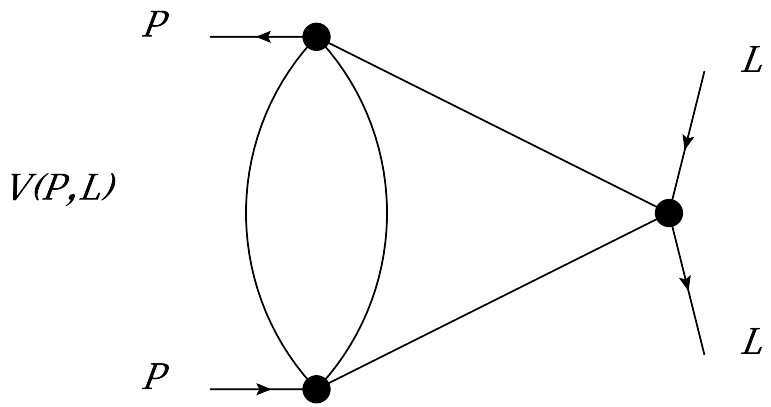
remaining term $I_1(V_0,V_0)$ is momentum independent

potential problem

for $\Delta_L = f(\{P_i\}, L) - f(0, L) \sim \frac{1}{L}\, f(\{P_i\})$

need right leading asymptotic behaviour

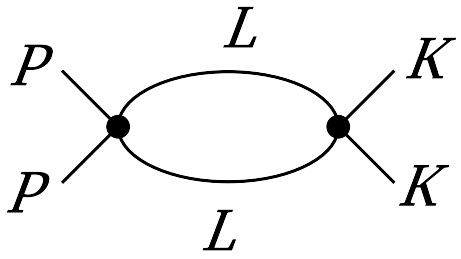
example:



for diagonal $V(P, L) - V(0, L) \nrightarrow \frac{1}{L}\, V(P)$

$\rightarrow$ can't regulate the L-loop by splitting an isolated diagonal V

example: diagram we can't regulate with vertex splitting:



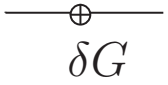
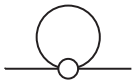
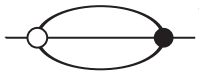
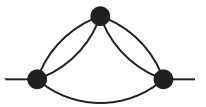
$$I_s = \int dL \underbrace{V(P,L)}_{\Uparrow} G^2(L) V(L,K)$$

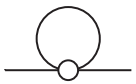
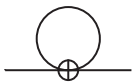
contains diagonal s–channel insertions

renormalization of the 2-point function (for now assume V finite)

KEY 1: eom for Σ simplifies to SD form using higher eom's

(MEC and Yun Guo, Phys. Rev. D83, 016006 (2011))

Σ = $-$  $+\frac{1}{2}$  $+\frac{1}{2}$  $+(2)\frac{1}{6}$  $+(2)\frac{1}{6}$  $-\frac{1}{6}$  $+\frac{1}{4}$ 

= $-$  $+\frac{1}{2}$  $+\frac{1}{2}$  $+\frac{1}{6}$  $+\frac{1}{6}$ 

$$\Sigma(p) = i(\delta Z P^2 - \delta m^2) + \frac{1}{2}(\lambda + \delta\lambda_{\rm et}) \int dQ\, G(Q)$$
$$+\frac{1}{6}(\lambda + \delta\lambda_{\rm bb}) \int dQ \int dK\, V(P,Q,K) G(Q) G(K) G(Q+K+P)$$

KEY 2: Σ eom can be repackaged to contain the Bethe-Salpeter vertex M

extract the leading asymptotic behaviour:

$$\tilde{G} = G_{\rm as} + \delta G\,, \quad \Sigma(\tilde{G}) = \Sigma_{\rm as} + \Sigma_0$$

where $G_{\rm as} \sim P^{-2}$ and $\Sigma_{\rm as} \sim P^2$

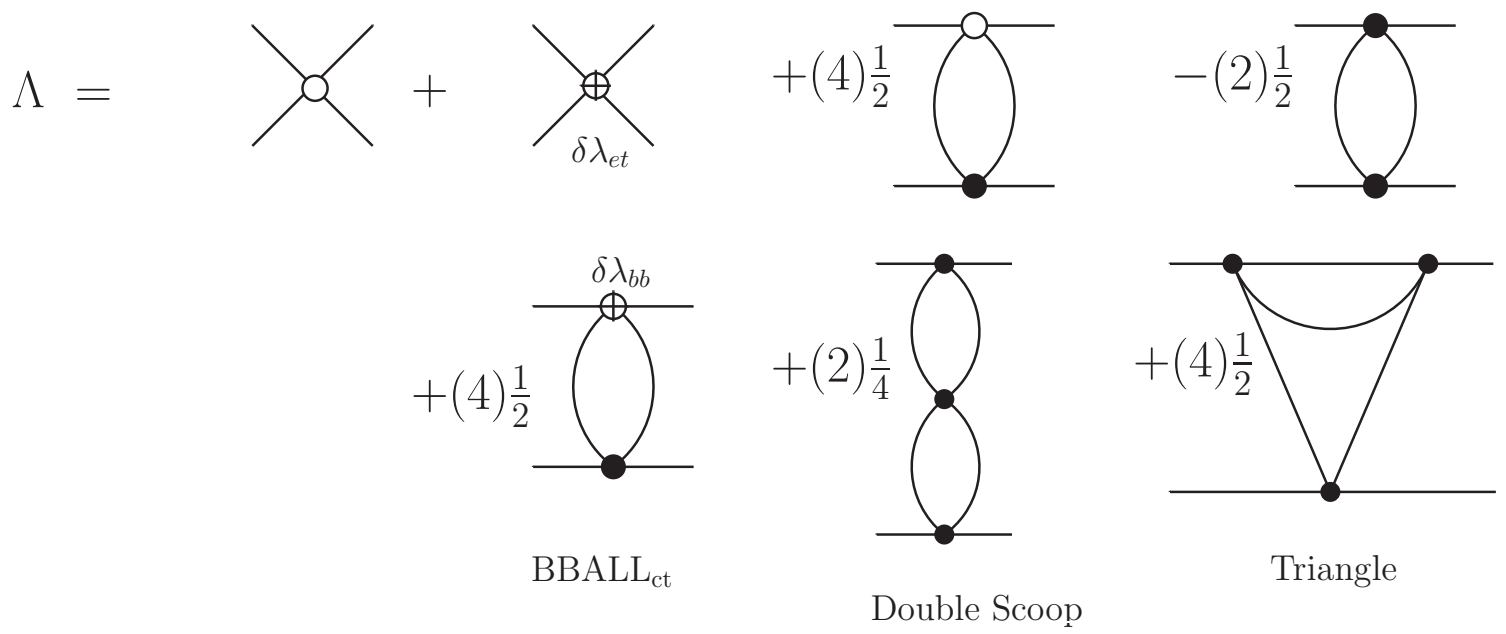
$$\Sigma_0(K) = -i\delta m^2 - \frac{i}{2}(m^2 + \delta m^2) \int dP G^2_{\rm as}(P) M(P,0) + \text{ finite terms}$$

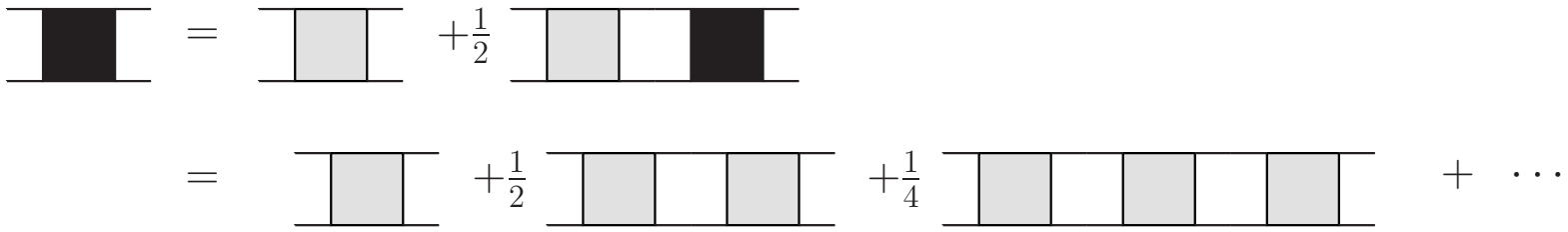
P-integral is divergent but $\vec{p}$-independent

$\Rightarrow$ can be cancelled by an appropriate choice of the counter-term δm^2

$\Rightarrow$ must show that the vertex M can be renormalized

Diagrams in Λ and M





KEY: Λ contains only t- and u-channel diagrams

consequences:

1: no diagonal V's in Λ or M - *from figure*

2: can use vertex splitting procedure to divide Λ into:

finite + (momentum-dependent div = ct) + (infinite constant = ct)

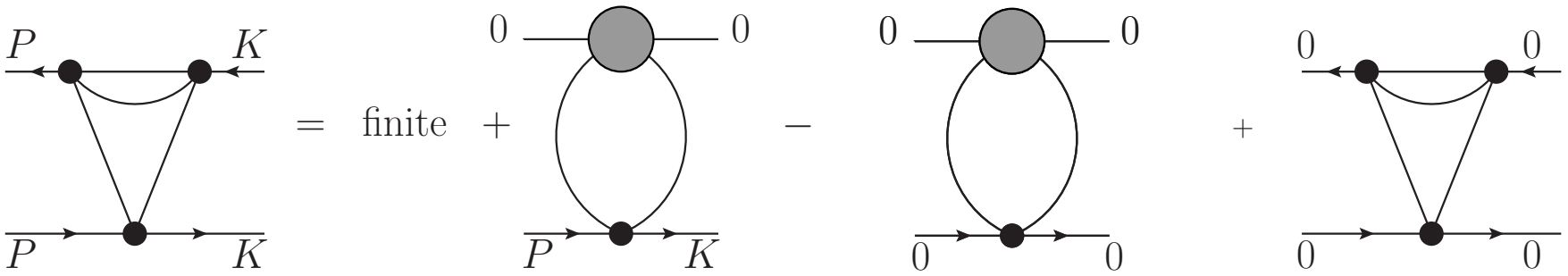
note: very non-trivial at 2-loops

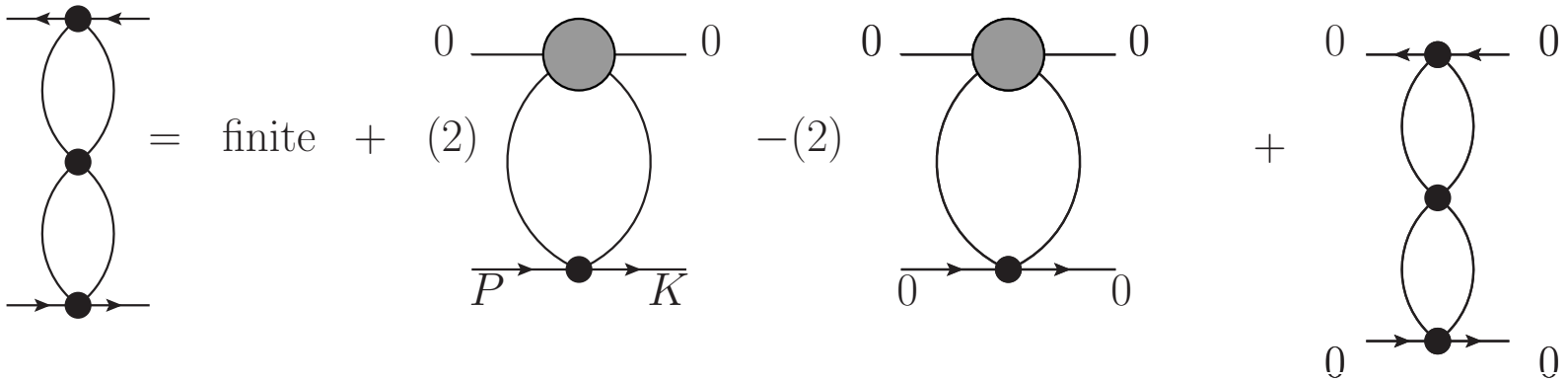
3: Λ itself can be split: $\Lambda(P,L) - \Lambda(0,L) \sim \frac{1}{L}\Lambda(P)$

4: can renormalize M $\rightarrow$ can renormalize Σ

2 - vertex splitting on double-scoop and triangle graphs

$\longrightarrow$ isolate overlapping divergences and $\vec{p}$-independent piece





use $\delta\lambda_{bb} = -\frac{3}{2} I_1(V_0, V_0)$ to cancel overlapping divergences

remaining infinite constant (partially) cancelled with $\delta\lambda_{et}$

$$\delta\lambda_{et} = \delta\hat{\lambda}_{et} + \Delta\lambda_{et} \quad \rightarrow \quad \Lambda = \Lambda^f + \Delta\lambda_{et}$$

$$\delta\hat{\lambda}^{(1)}_{et} = -2I_1(\lambda, V_0) + I_1(V_0, V_0)$$

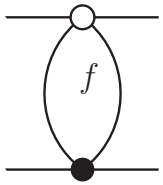
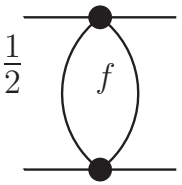
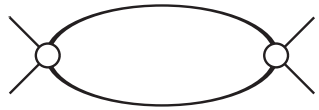
$$\delta\hat{\lambda}^{(2)}_{\rm et} = -\frac{1}{2}I_2(V_0, V, V_0) - 2I_3(V_0, V, V_0) + 3\frac{1}{\lambda}I_1(\lambda, V_0)I_1(V_0, V_0)$$

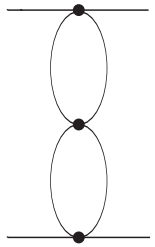
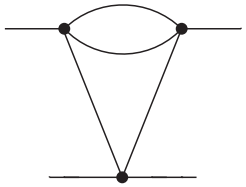
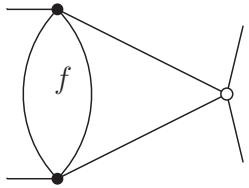
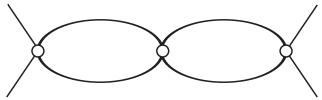
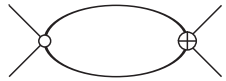
3 - $\Lambda_f(P, K)$ is finite and contains only t and u (not s) channels

$\rightarrow \Lambda_f(P, L) - \Lambda_f(0, L) \sim \frac{1}{L}\Lambda(P)$

4 - Can renormalize M (and Σ)

- M contains no new sub-divergences from cutting through kernels
  or: *only get divergences from loops connecting the Λ's*
- all diagonal vertices are bare

$M^{(1)} = \Delta\lambda_{et}^{(1)} + (4)\frac{1}{2}$  $-(2)\frac{1}{2}$  $+\frac{1}{2}$ 

$M^{(2)} = \cdots +$  finite  finite    $\Uparrow \Delta\lambda_{et}^{(1)}$  $\Delta\lambda_{et}^{(2)}$

- construct $\Delta\lambda_{et}$ to cancel divergences from iterating in M eqn

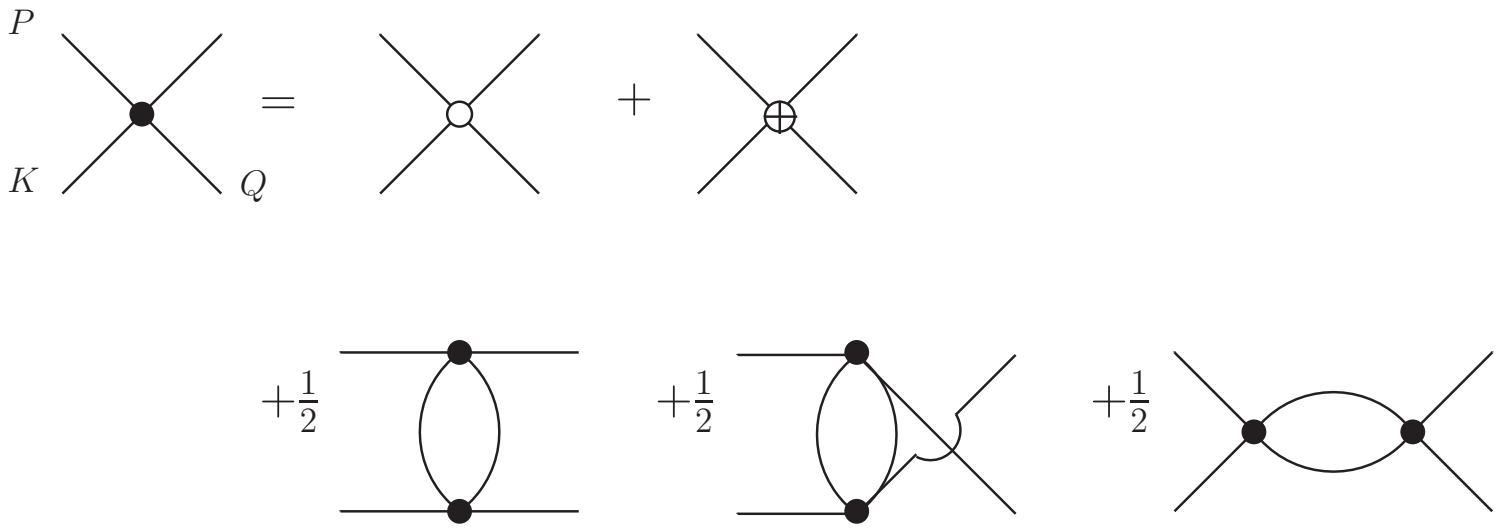
can show that M is renormalized using $\Delta\lambda_{\rm et}$ determined self-consistently:

$$M(P,K) = [\Lambda_f(P,K) + \Delta\lambda_{\rm et}] + \frac{1}{2}\int dR\,[\Lambda_f(P,R) + \Delta\lambda_{\rm et}]G(R)^2 M(R,K)$$

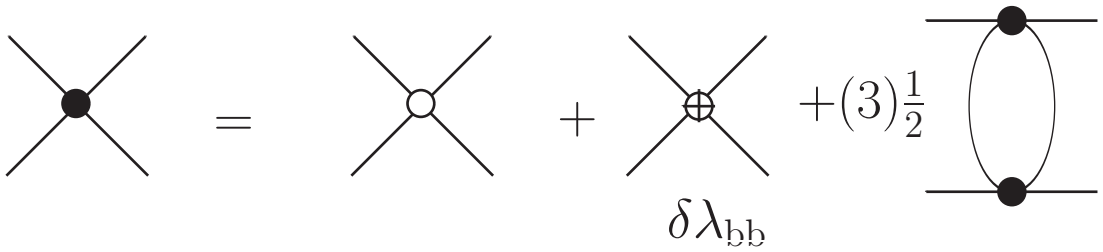
$$\Delta\lambda_{\rm et} = -\frac{1}{2}\int dL\,[\Lambda_f(0,L) + \Delta\lambda_{\rm et}]\,G^2(L)M(L,0)$$

Renormalization of V:

V obtained from eom:



short-hand notation:



vertex splitting gives same counter-term: $\delta\lambda^{(1)}_{\rm bb} = -\frac{3}{2} I_1(V_0, V_0)$

however: must show can cancel sub-divergences in V

consider the perturbative expansion of the V eom

general result:
expansion of the eom for the j-point function from the nPI effective action
$\longrightarrow$ matches the expansion of the full function to order $n - j + 1$
(MEC and Yun Guo, Phys. Rev. D83, 016006 (2011))

- the 4-point function from the 4PI effective action matches to 1-loop
$\Rightarrow \delta\lambda_{\rm bb}$ has non-pert contribution at > 2 loops in skeleton exp (like $\Delta\lambda_{\rm et}$)

don't know analytic form

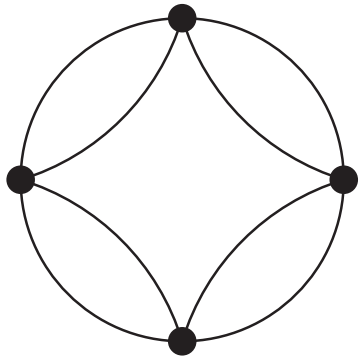
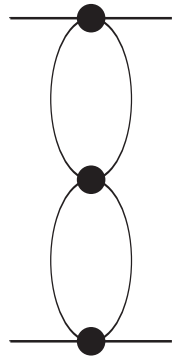
check minimum requirement:
skeleton exps of $\delta\lambda_{\rm et}$ and $\delta\lambda_{\rm bb}$ must be same to order of the truncation

at 1-loop can easily see this is true - *not true at 2-loops*

reason: Λ and V produced at different orders in the skeleton expansion

$\delta/\delta G$ opens 1-loop
$\delta/\delta V$ opens 3-loops

can show: include contributions from 5-loop terms in effective action to Veom

$-\frac{1}{128}$  $\Rightarrow$ $-(3)\frac{1}{4}$ 

$\Rightarrow$ counter-terms $\delta\lambda_{\rm bb}$ and $\delta\lambda_{\rm et}$ are the same to 2-loops

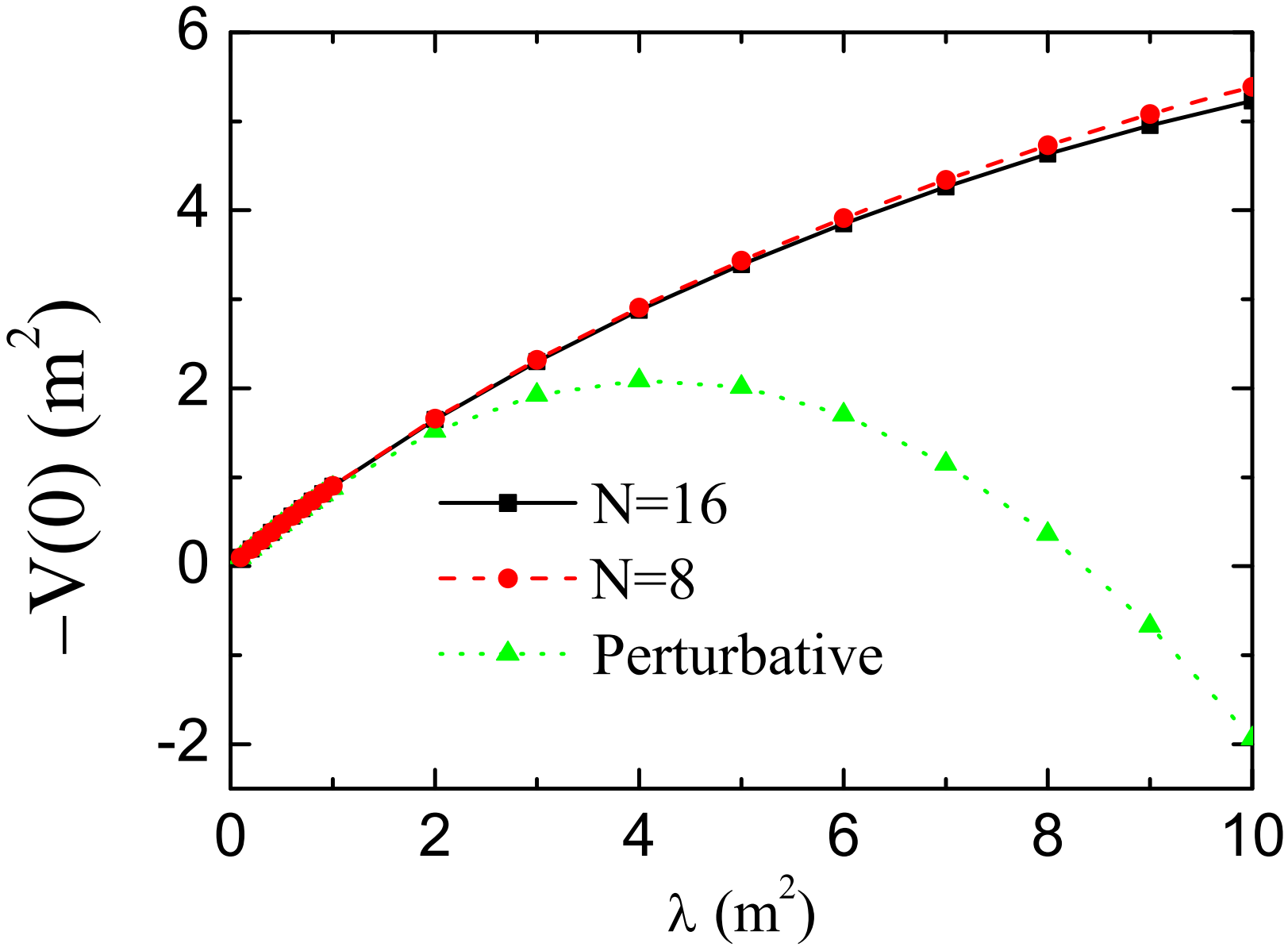
Numerical results:

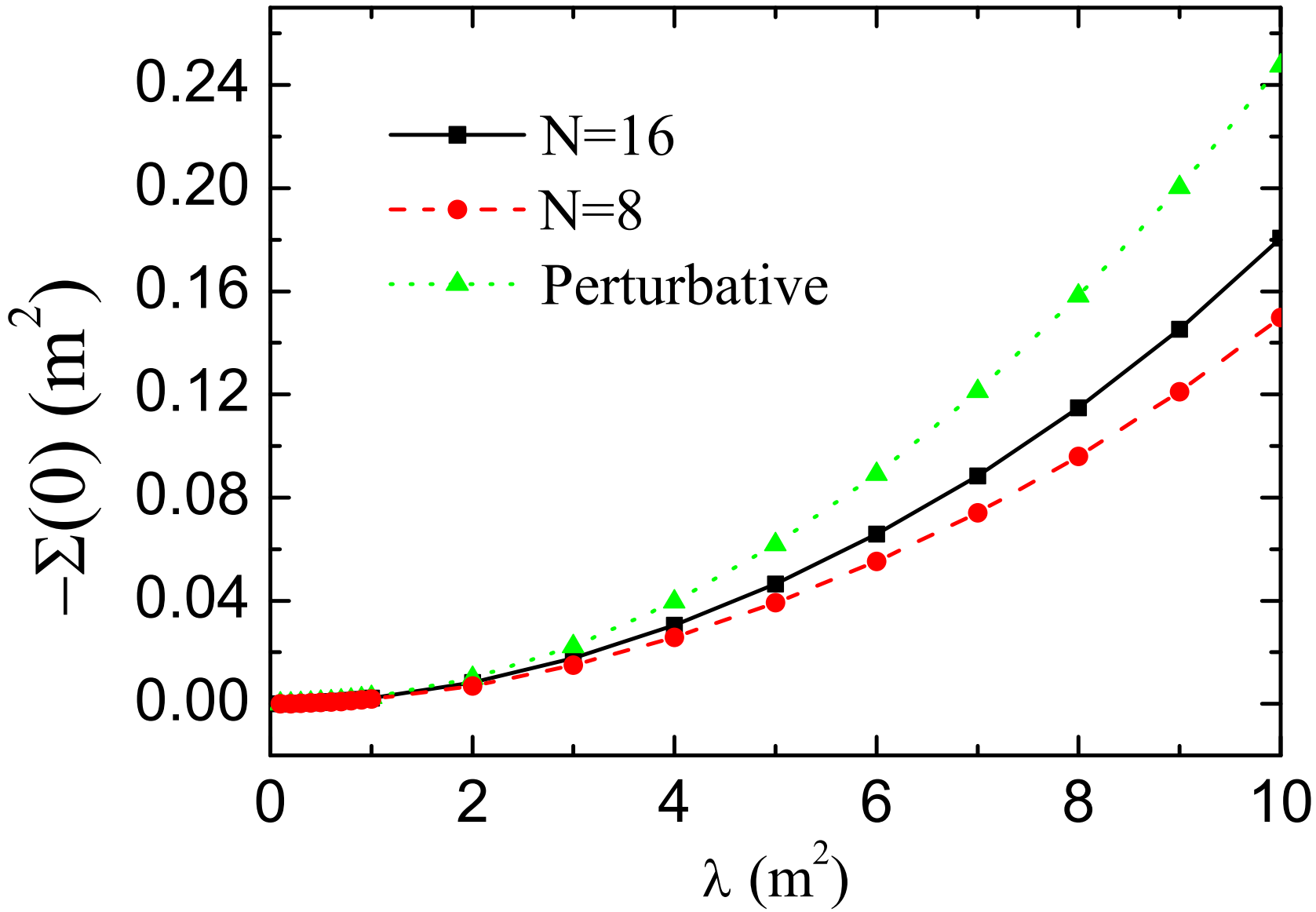
work in 2-dimensions

only divergence is a log divergence from tadpole diagram in 2-point function

absorb with a counter-term δm^2

$N \times N$ 2-d symmetric lattice, periodic bc, lattice spacing $a = 2\pi/(Nm)$


6
4
2
0
-2
0
2
4
6
8
10
$-V(0)\ (m^2)$
$\lambda\ (m^2)$
N=16
N=8
Perturbative


0.24
0.20
0.16
0.12
0.08
0.04
0.00
$-\Sigma(0)\,(m^2)$
N=16
N=8
Perturbative
0
2
4
6
8
10
$\lambda\,(m^2)$

Conclusions:

- 4PI effective action has the right structure for renormalization
- numerical calculations are hard but preliminary results are promising
- need to do $T \neq 0$ calculation to really see if 4PI gives an advantage