

O(2) quantum theory at finite densities and the generalised density of states

SEWM, 10.7.2012

Kurt Langfeld

School of Comp. and Mathematics and The HPCC, Univ. of Plymouth, UK

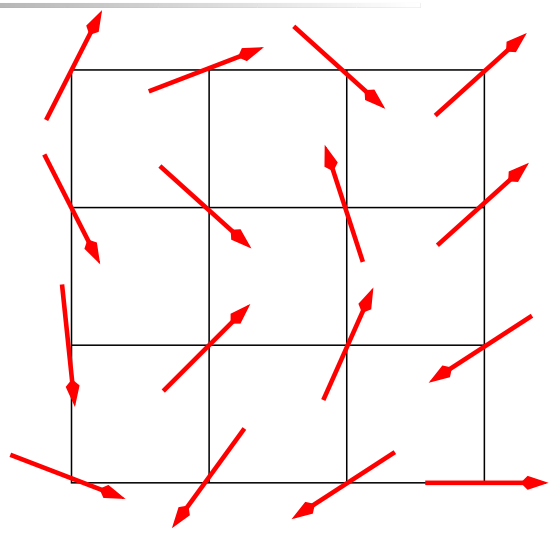
with:

Biagio Lucini, Swansea

Antonio Rago, Plymouth

Roberto Pellegrini, Turino

The O(2) model - basic setup



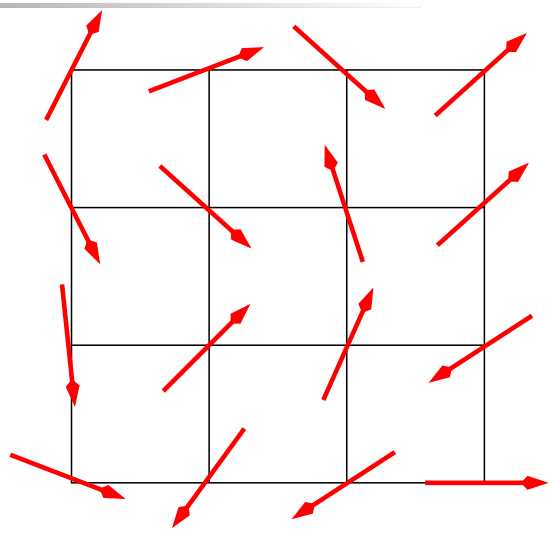
- Degrees of freedom:

  $\vec{n}_x, \quad \vec{n}^2 = 1, \quad \vec{n}_x = (\cos\phi_x, \sin\phi_x)^T$

  Global O(2) symmetry:

  $\vec{n}_x \rightarrow O\,\vec{n}_x, \quad O \in O(2)$

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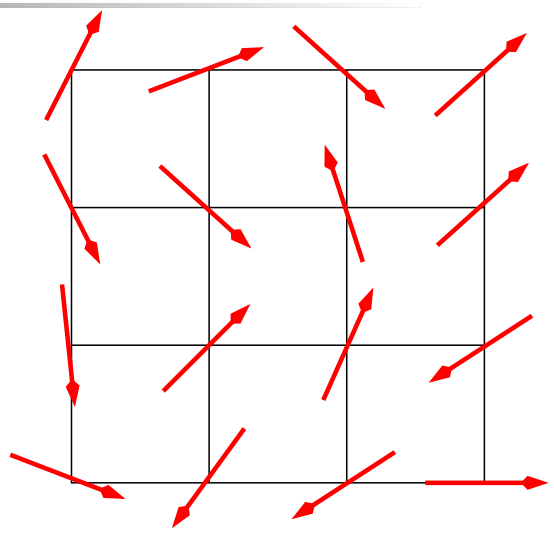
- Partition function: $Z = \int \mathcal{D}\phi_x \, \exp\{\beta S[\phi]\}$

  action: $S[\phi] \;=\; \sum_{\langle xy \rangle} \vec{n}_x \cdot \vec{n}_y \;=\; \sum_{\langle xy \rangle} \cos(\phi_x - \phi_y)$.

  (also called the xy-model)

  β : coupling constant *OR* inverse temperature

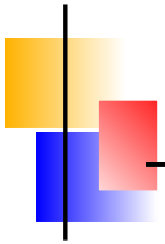
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- What do we know about this model?

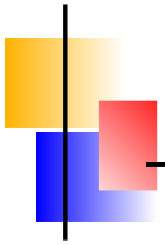


The O(2) model - first results

- Very good alogorithms to simulate the O(2) model:
  - △ Wolff-cluster algorithm: [Wolff, NPB 322 (1990) 581].
  - △ : worm algorithm (see later)

    [Prokof'ev, Svistuno, PRL 87 (2001) 160601]

    [Banerjee, Chandrasekharan, PRD 81 (2010) 125007]
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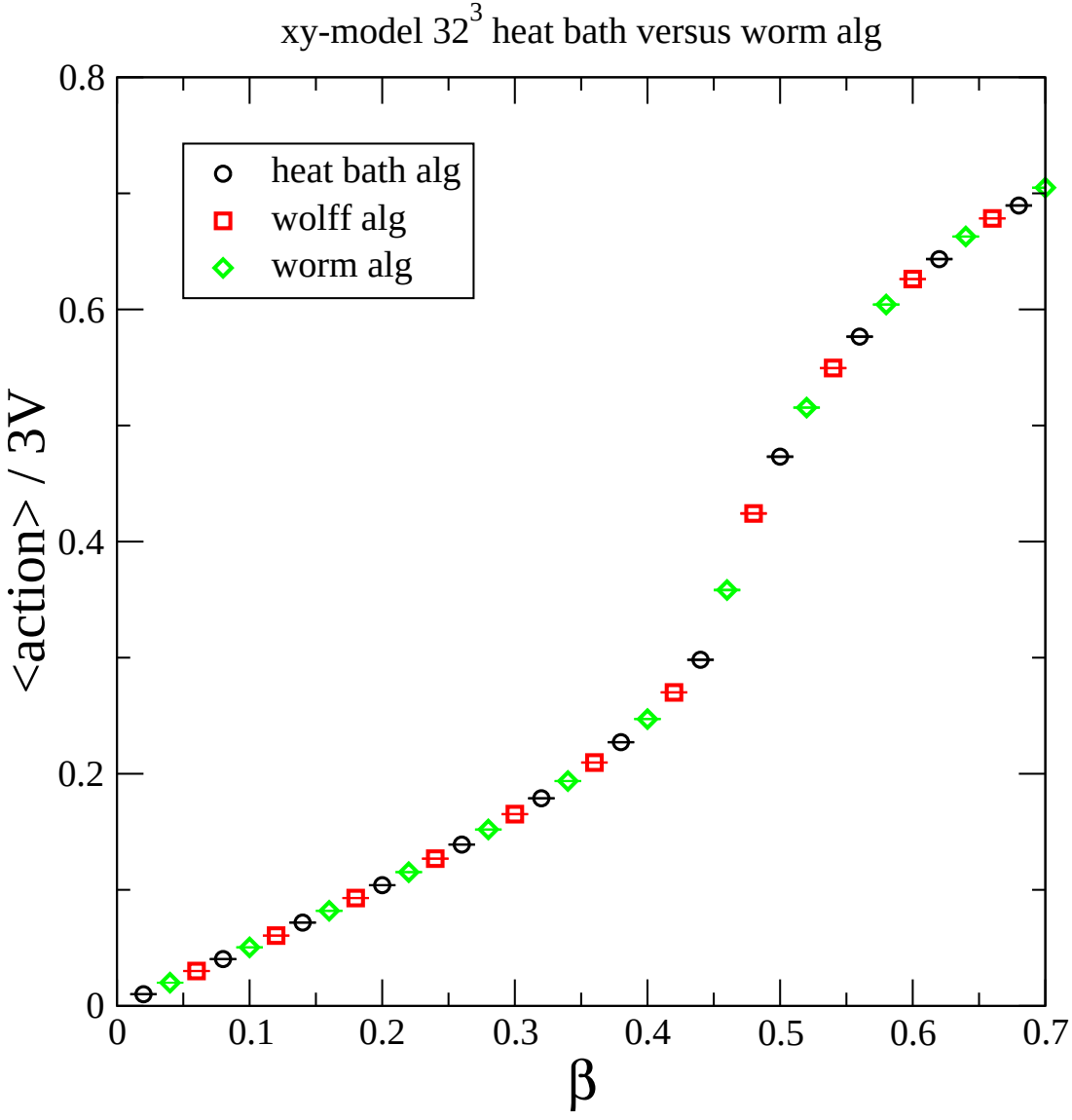
- Observables:

  $\Rightarrow$ action density as a function of β

  $\Rightarrow$ magnetic susceptibility $\chi(\beta)$

The O(2) model - first results

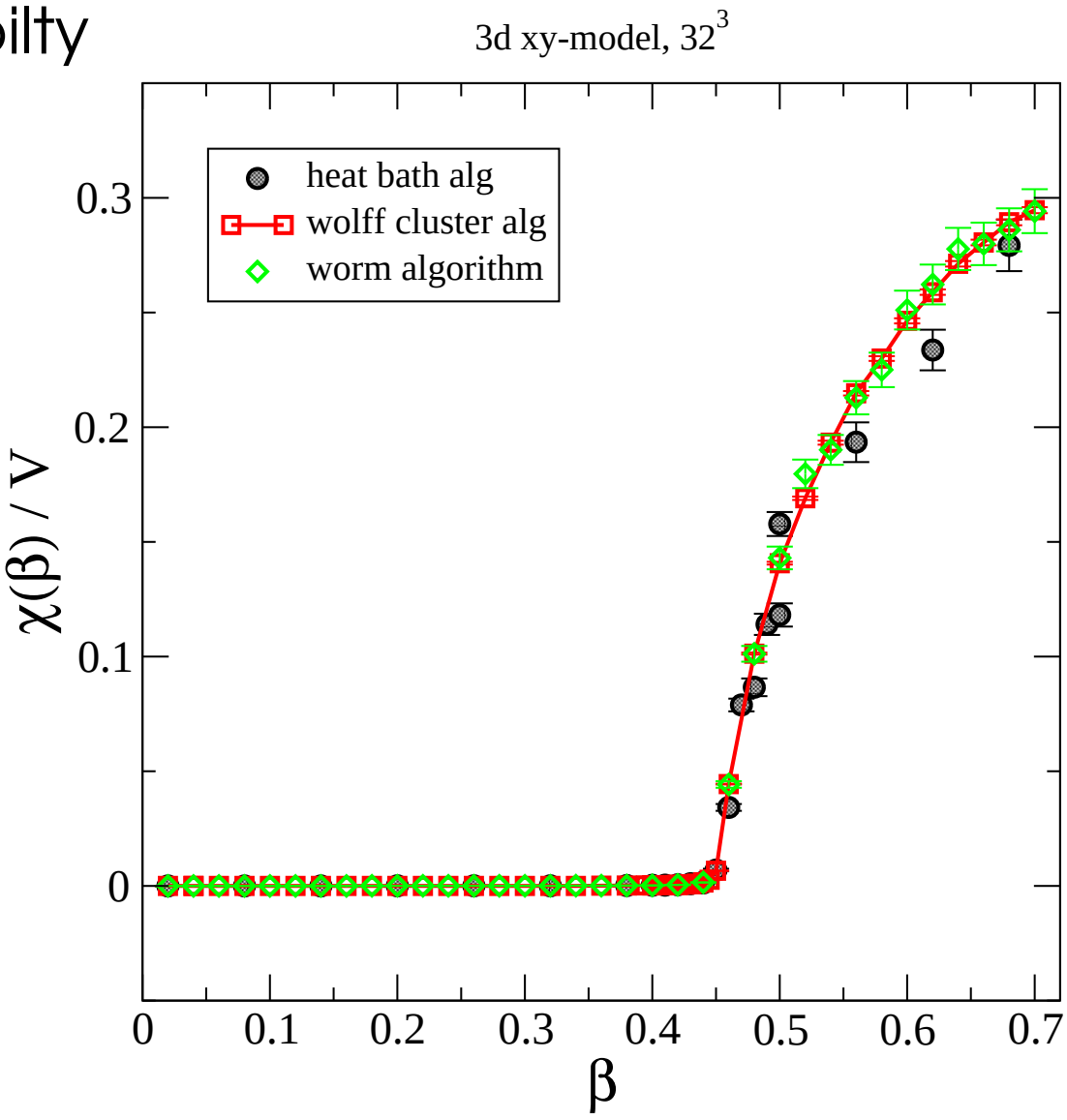
- action density:



[KL in preparation]

The O(2) model - first results

- magnetic susceptibilty *per volume*:



[KL in preparation]

The O(2) model - first results

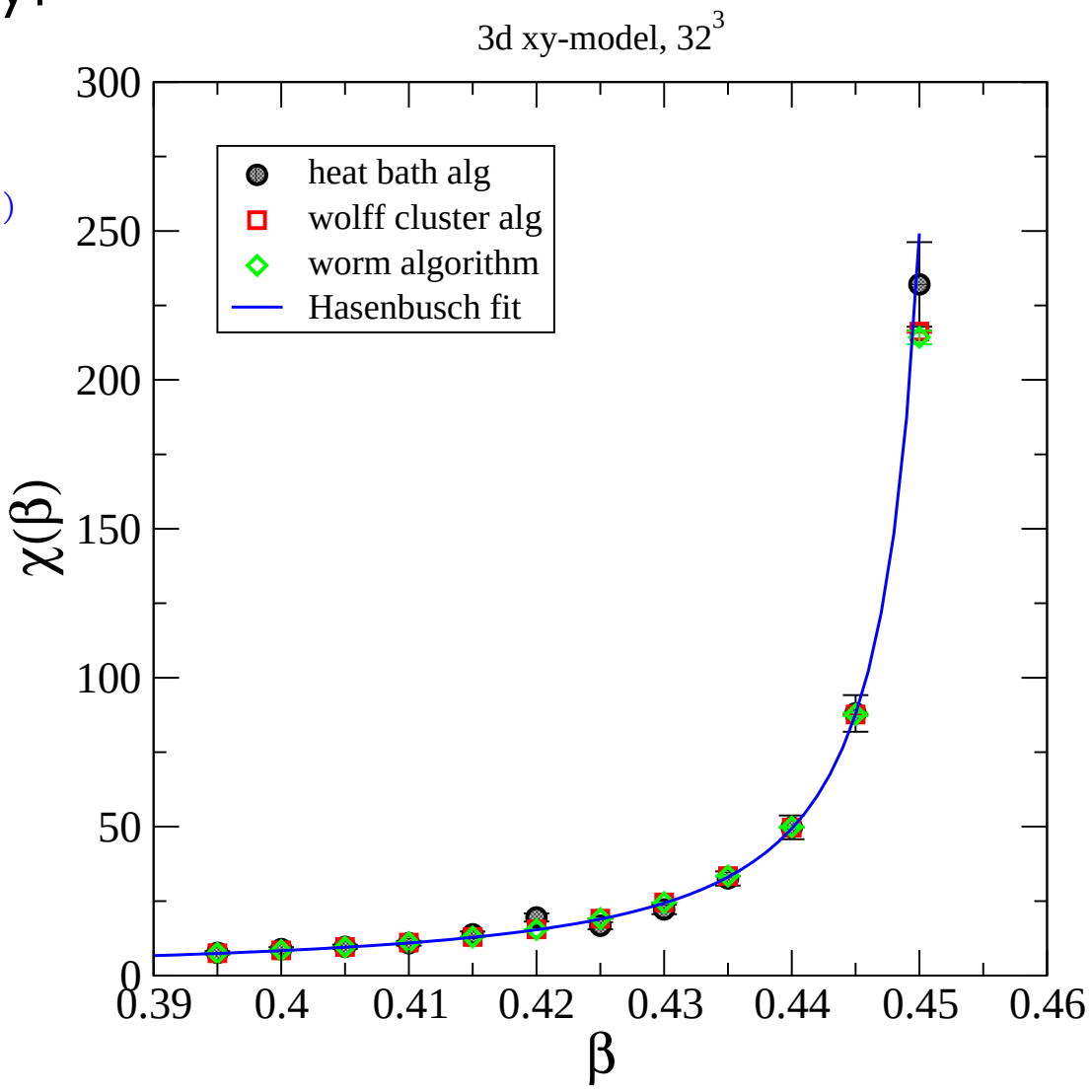
- magnetic susceptibility:

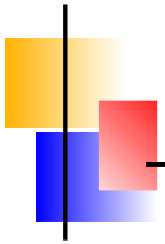
$\beta < \beta_c$

$\chi(\beta) \approx 0.5045(1)\left(1 - \frac{\beta}{\beta_c}\right)^{-1.324(1)}$

$\beta_c \approx 0.45417(2)$

[KL in preparation]



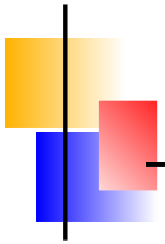


The O(2) model - finite densities

- Conserved Noether charge:

$$\Rightarrow \; j_\mu(x) \propto \sin\Big(\phi_x - \phi_{x+\mu}\Big)$$

indeed: $\sum_\mu \triangle_\mu j_\mu := \sum_\mu \Big[j_\mu(x) - j_\mu(x-\mu) \Big] \; = \; 0.$



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- Finite densities: introduce chemical potential μ (couples to the Noether charge):

  $S \; = \; \sum_{x,\mu} \cos\Big(\phi_x - \phi_{x+\mu} \; + \; i\mu\delta_{\mu 0}\Big)$

The O(2) model - finite densities

- ....but:

$\mu \neq 0$ ⟶ sign problem ⟶ ~~cluster alg~~

~~heat bath alg~~

The O(2) model - finite densities

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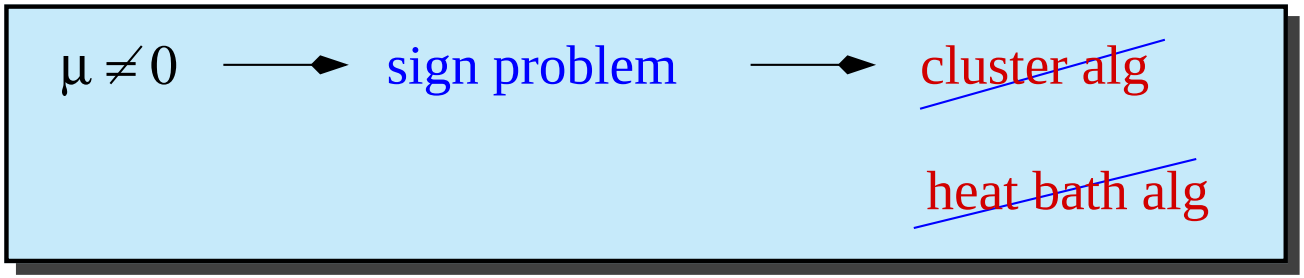
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- It is time to look at the worm algorithmus!

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- use: $\exp\{\beta\,\cos\phi\} \;=\; \sum_k I_k(\beta)\,\mathrm{e}^{ik\phi}$
  to integrate out the fields ϕ_x

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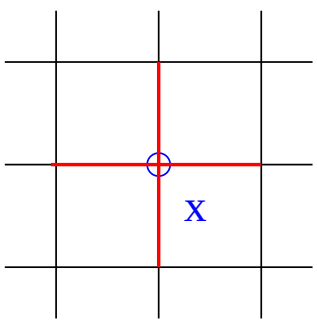
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- It is time to look at the worm algorithmus!
- use: $\exp\{\beta\ \cos\phi\} \ = \ \sum_k I_k(\beta)\, \mathrm{e}^{ik\phi}$
  to integrate out the fields ϕ_x
- Partition function: $Z = \sum_{\{k_\ell\}} \prod_\ell I_{k_\ell}(\beta)\, \mathrm{e}^{\mu k_\ell \delta_{\nu 0}}\ \delta\left(\sum_{\ell \in x} k_\ell\right)$

  for each link ℓ: $k_\ell = .. -2, -1, 0, 1, 2..$

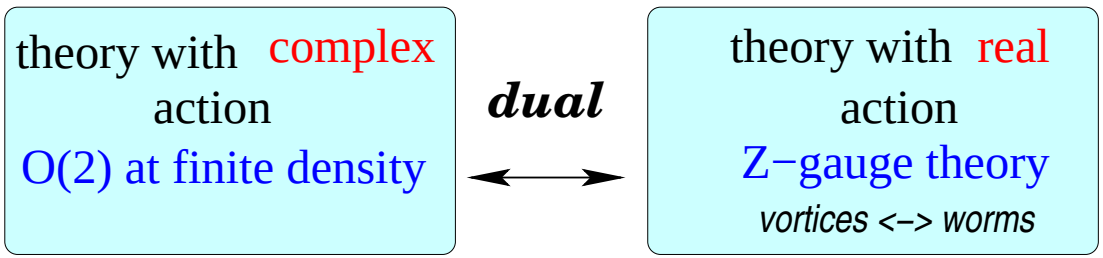
  $\ell \in x$ addresses all links ℓ

  attached to x



The O(2) model - finite densities

- How do we solve the sign problem here?



The O(2) model - finite densities

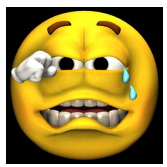
- How do we solve the sign problem here?

theory with complex action
O(2) at finite density

dual ⟷

theory with real action
Z−gauge theory
vortices <–> worms

- finite density results ⇒ later



The O(2) model - finite densities

- How do we solve the sign problem here?

theory with complex action O(2) at finite density	***dual*** ⟷	theory with real action Z−gauge theory *vortices <–> worms*

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- so far: statistical field theory

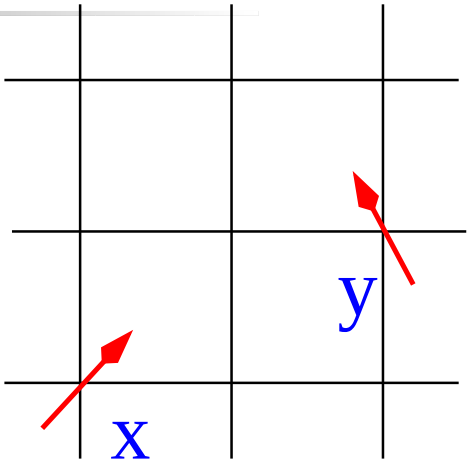
  now:

Quantum Field Theory

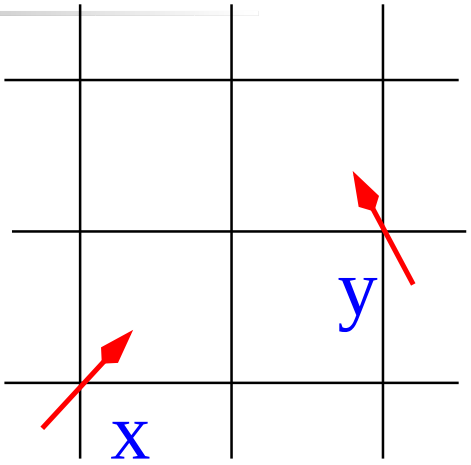
The O(2) model - QFT rising

- correlation length $\xi(\beta)$
  Green function:

$$G(x-y) = \langle \vec{n}_x \cdot \vec{n}_y \rangle \propto \exp\{-|x-y|/\xi\}$$



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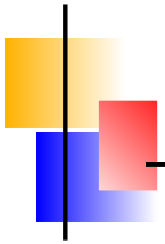
- phase transition: $\xi(\beta_c) \to \infty$

$$\xi(\beta) = \xi_0 \left(1 - \frac{\beta}{\beta_c}\right)^{-\nu}$$

Use the Wolff cluster algorithm
with an improved esitmator [Wolff, NPB 322 (1990) 581]

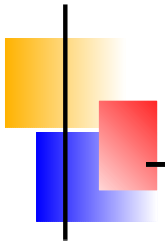
$$\xi_0 = 0.4866(26), \quad \nu = 0.6711(16), \quad \beta_c = 0.454157(14)$$

[Hasenbusch, NPB 333 (1990) 581]



The O(2) model - QFT rising

- Statistical field theory:
  correlation length $\xi(\beta)$ changes
  with the inverse temperature β



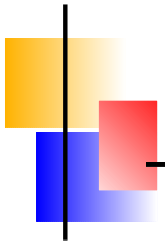
The O(2) model - QFT rising

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- Quantum field theory:
  mass gap m:

$$ma(\beta) = 1/\xi(\beta)$$

  lattice spacing $a(\beta)$ changes
  with coupling β



The O(2) model - QFT rising

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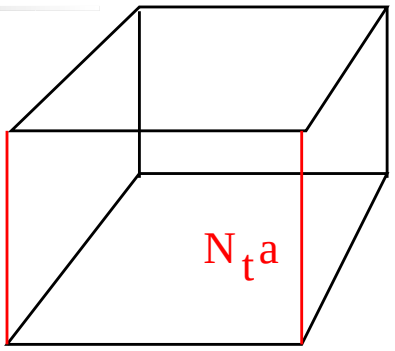
  lattice spacing $a(\beta)$ changes
  with coupling β

- continuum limit: $\beta \to \beta_c$, m : fixed, $a(\beta) \to 0$
  `[dimensional transmutation]`

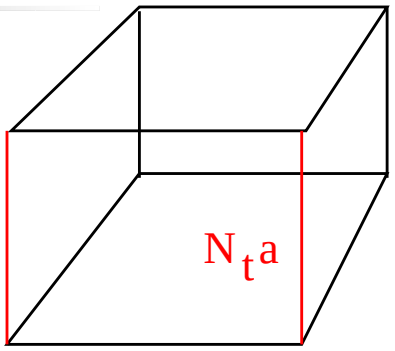
The O(2) model - QFT rising

Observables:

- Temperature: $T = 1/N_t a \Rightarrow T/m = \xi(\beta)/N_t$



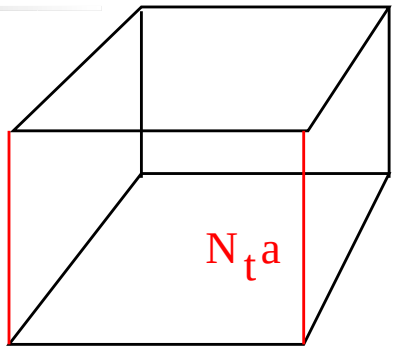
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The O(2) model - QFT rising



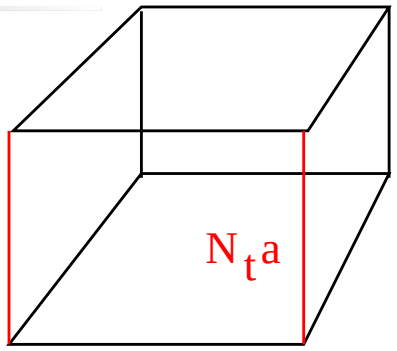
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- Magnetic susceptibility: $\chi/m^2 = Z_\chi(\beta)\chi_{\rm latt}\, \xi^2(\beta)$

  $Z_\chi(\beta)$: `wave function renormalisation`

  `use:` $\chi(T)/\chi(T=0)$

The O(2) model - QFT rising



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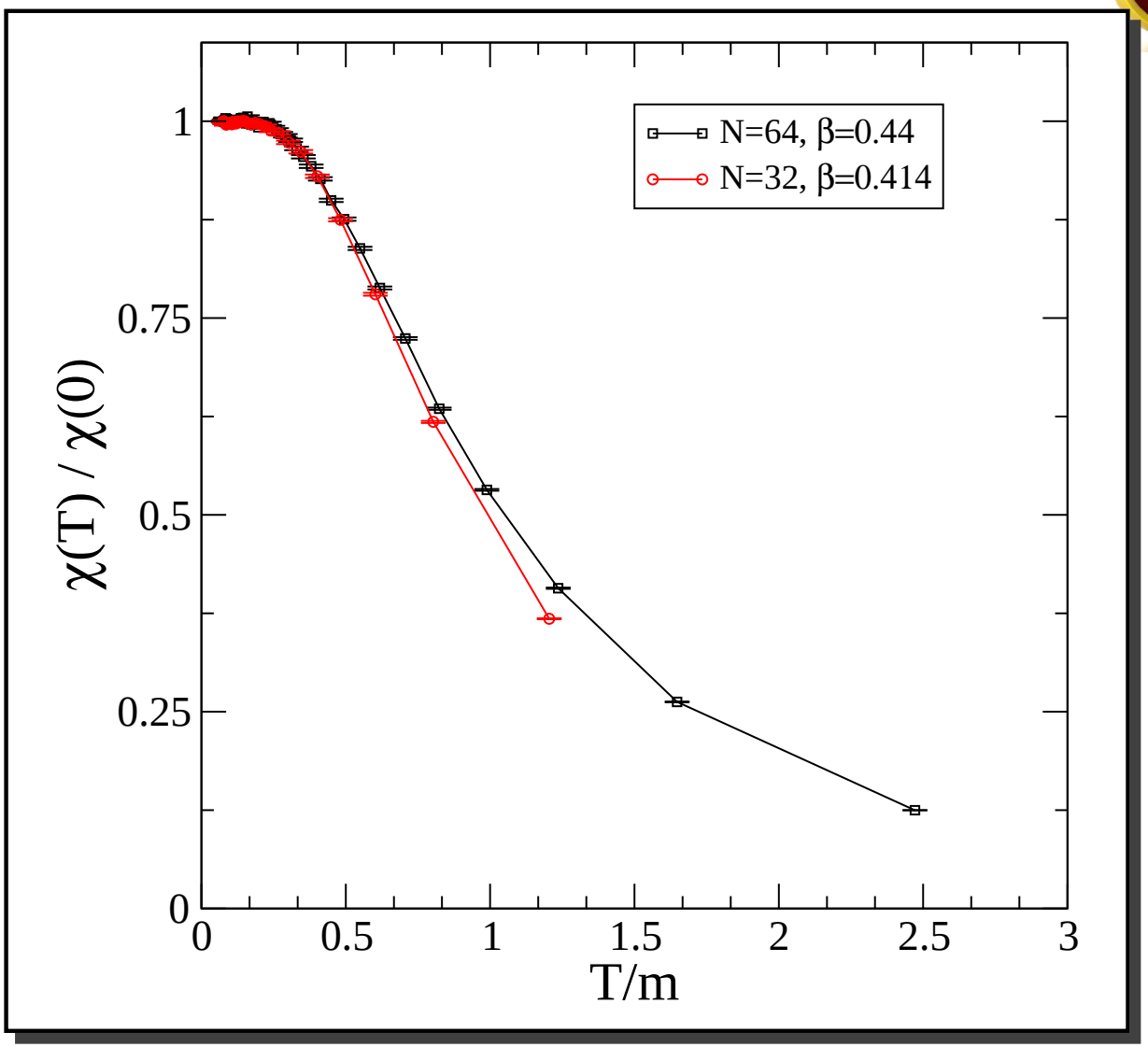
  `use:` $\chi(T)/\chi(T=0)$
- Particle density: $\rho/m^2 = \rho_{\rm latt}\, \xi^2(\beta)$

  `no wave function renormalisation!`

  (conserved current)

The O(2) model - QFT rising

Results - magnetic susceptibility

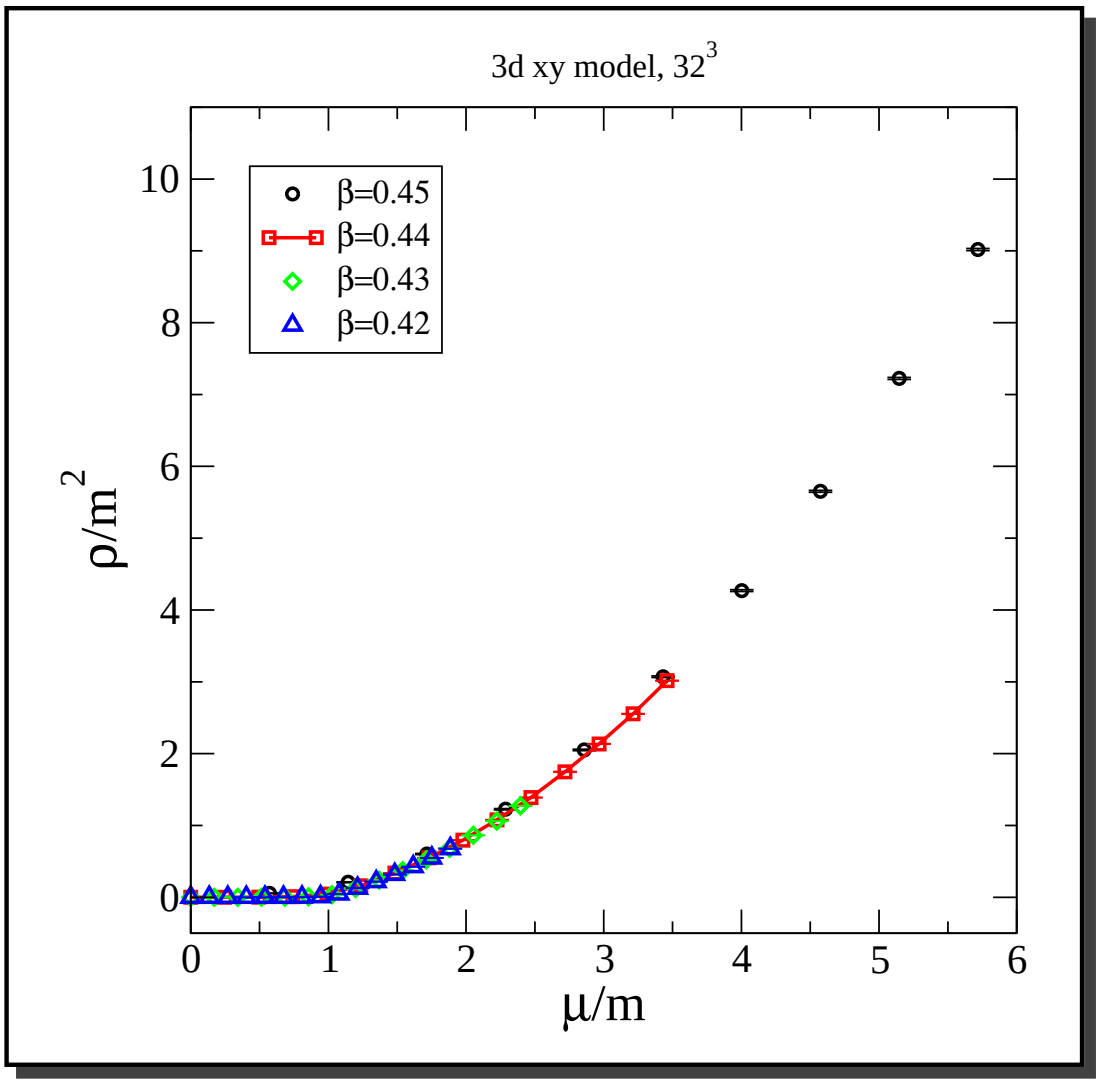


zero density ($\mu = 0$)

finite temperatures

The O(2) model - QFT rising

Results - particle density



$\mu_{\rm onset} = m$

no silver-blaze problem!

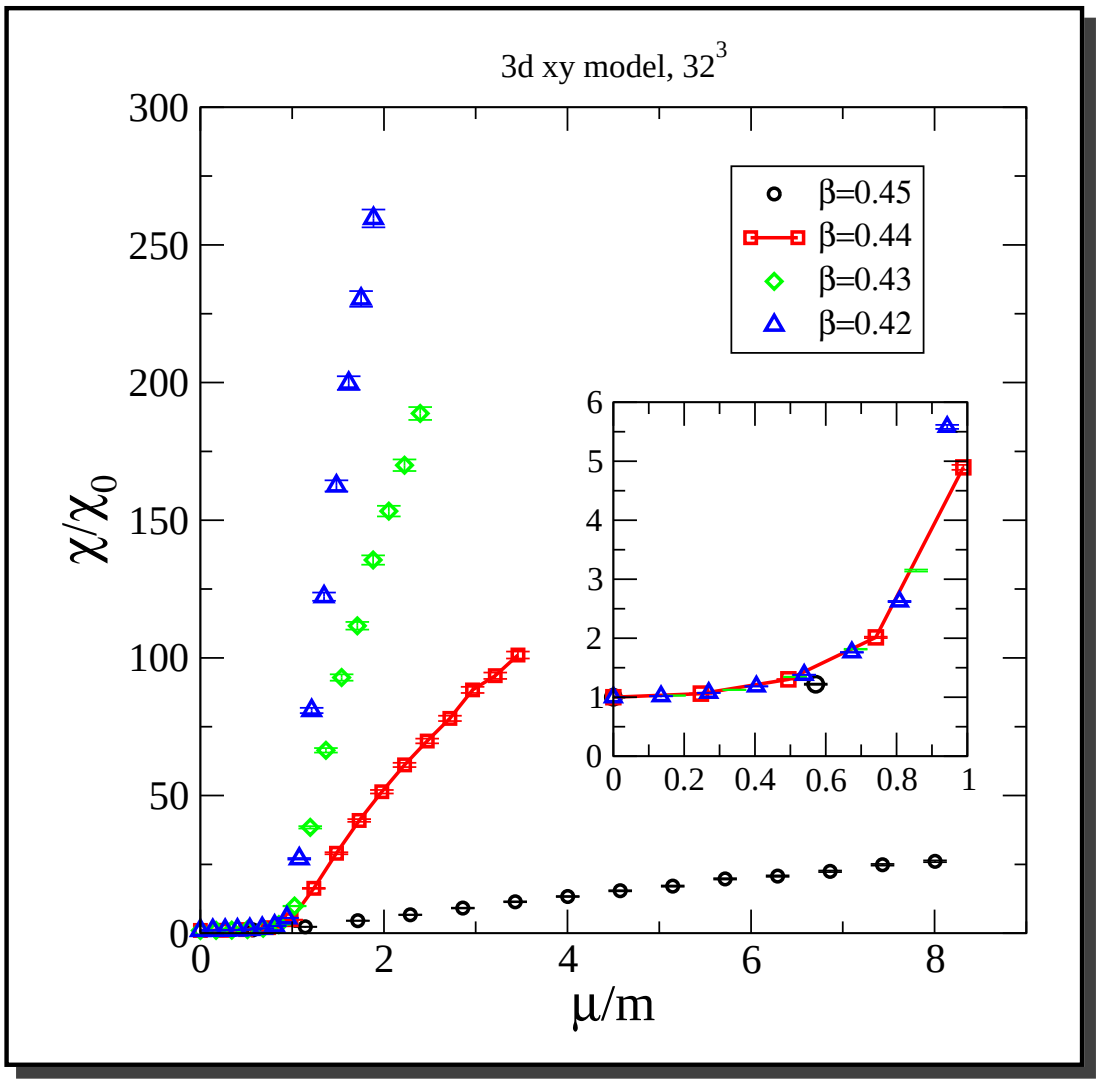
no wave function renorm.!

no Bose singularity !

[KL in preparation]

The O(2) model - QFT rising

Results - magnetic susceptibility agai



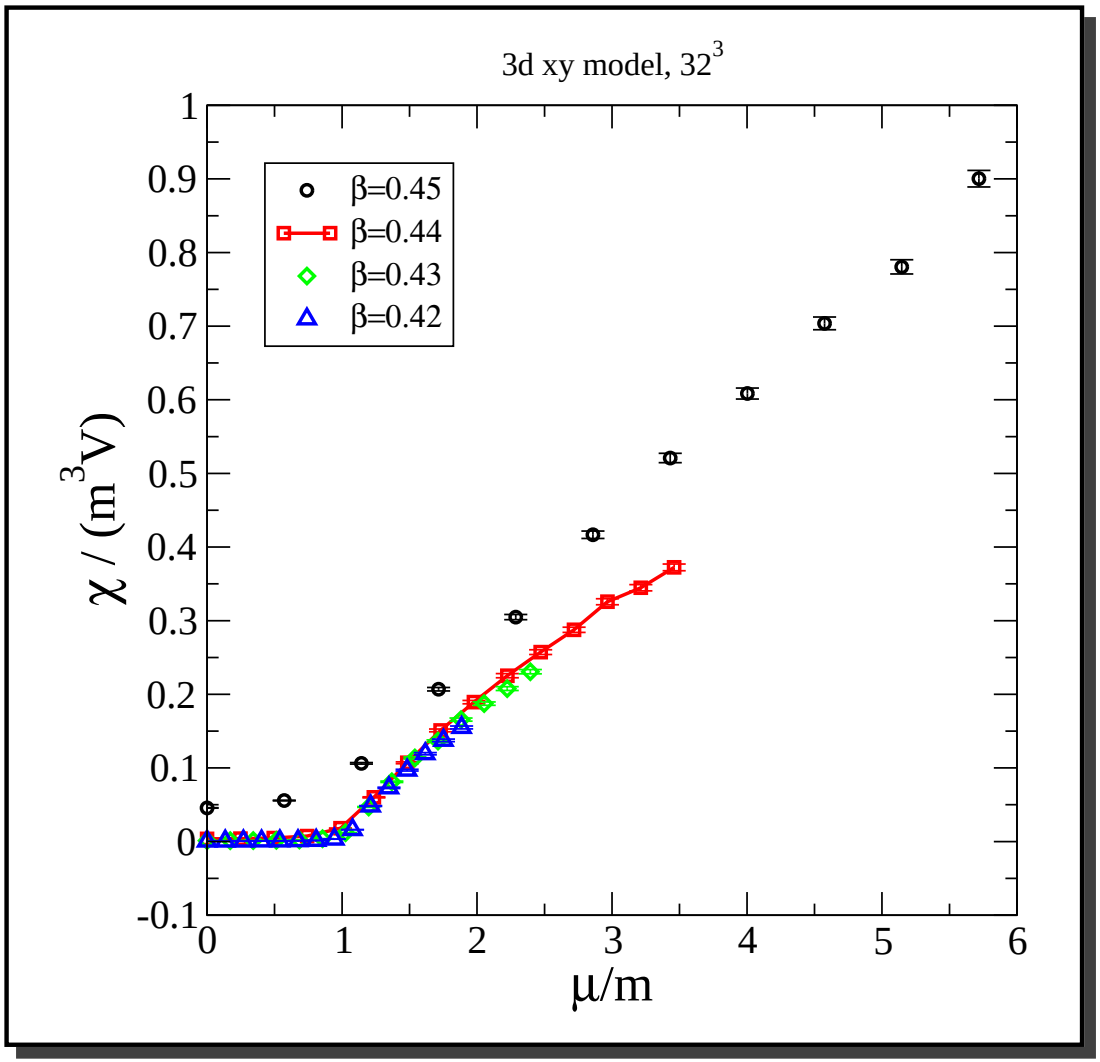
zero temperature

$\mu > m??$

[KL in preparation]

The O(2) model - QFT rising

Results - magnetic susceptibility agai



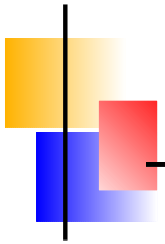
χ per volume!

$\mu > m$ SSB!

chemical catalysis!

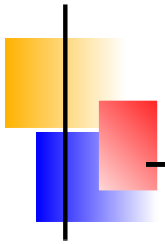
[presence of charges modify interactions $\Rightarrow$ SSB]

[KL in preparation]



Solving sign-problems

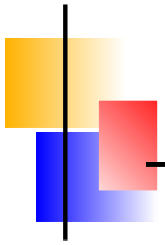
Can we simulate the O(2) model directly rather than its dual?



The density of states

- What is the density of states? `[my definition]`
  start with a partition function: $Z = \int \mathcal{D}\phi \, \exp\{\beta S[\phi]\}$

  β : coupling constant (QFT), inverse temperature (solid state physics)



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- Define the density of states $\rho(E)$ by:

$$\rho(E) \; = \; \int \mathcal{D}\phi \, \delta\Big(E - S[\phi]\Big)$$

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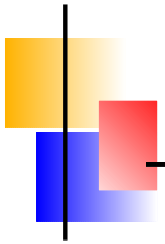
- What can we do with $\rho(E)$?

  Get the partition function for all β: $Z = \int dE \, \rho(E) \, \mathrm{e}^{\beta E}$

  `[scaling analysis in QFT]`

  Get action observables: $\langle f(S[\phi]) \rangle = \frac{1}{Z} \int dE \, \rho(E) \, f(E) \, \mathrm{e}^{\beta E}$

  e.g. specific heat: $\langle S^2 \rangle - \langle S \rangle^2$



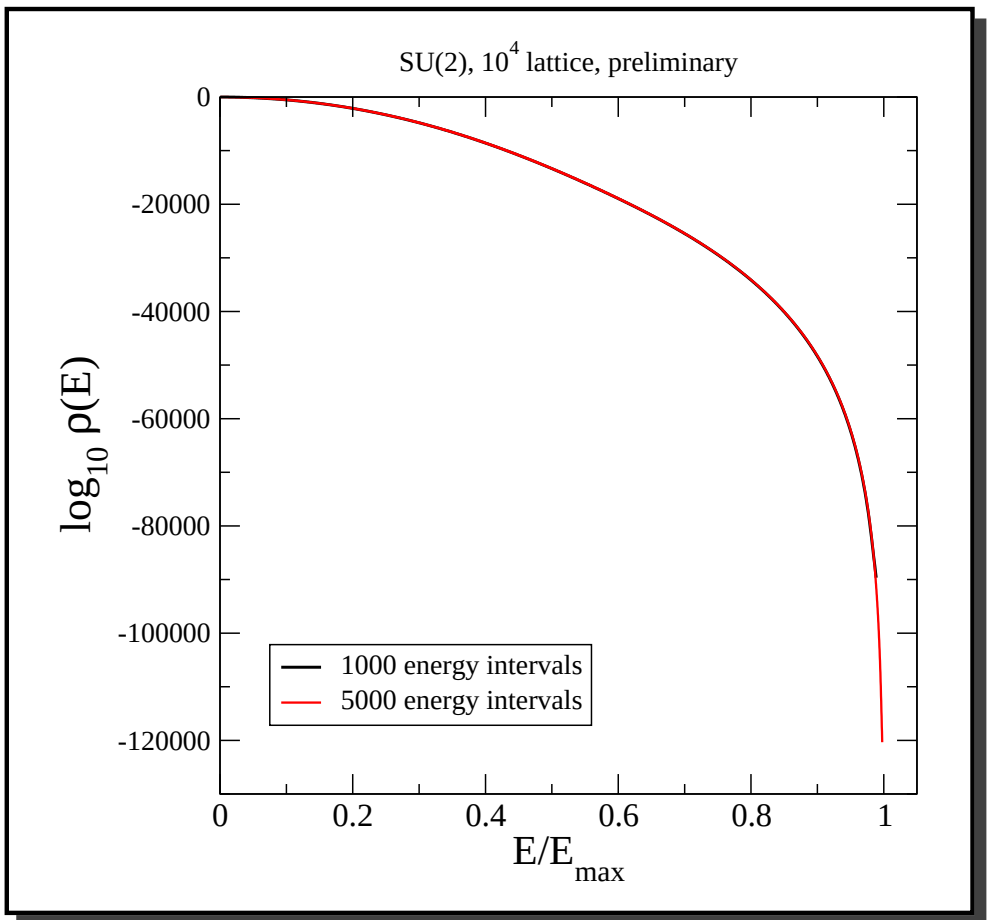
The density of states

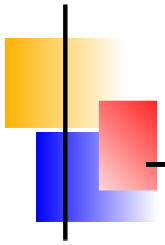
- How do we calculate $\rho(E)$ numerically?

The density of states

- How do we calculate $\rho(E)$ numerically?
- Observation:
  $\log \rho(E)$ is a *remarkable* smooth function of E!

example:
SU(2), 10^4 lattice:
$E_{\max} = 60,000$:



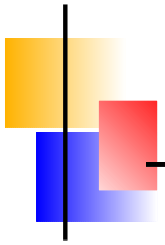


The density of states

- Choose a piecewiese linear ansatz:

$$\rho(E) = \rho(E_0)\ \exp\Big\{a(E_0)\,(E - E_0)\Big\}, \quad E_0 < E < E_0 + \delta E$$

need to find the $a(E_0)$!



The density of states

- Choose a piecewise linear ansatz:

$$\rho(E) = \rho(E_0)\, \exp\Big\{a(E_0)\,(E - E_0)\Big\}, \quad E_0 < E < E_0 + \delta E$$

need to find the $a(E_0)$!

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- Strategy: truncation + reweighting define:

$$\langle\!\langle f(E) \rangle\!\rangle(a) = \frac{1}{\mathcal{N}} \int \mathrm{d}E\ f(E)\ \rho(E)\, \theta_{[E_0,\delta E]}\ \ \mathrm{e}^{-aE}$$

trunction: $\theta_{[E_0,\delta E]} \Rightarrow$ only energies from the interval $[E_0, \delta E]$

reweighting: $\rho(E)\mathrm{e}^{-aE} \Rightarrow$ flat distribution over energy interval !

The density of states - SU(2)

- ...and here it comes:



△ calculate $a(E_0)$
for all energies (of interest)
△ reconstruct $\rho(E)$

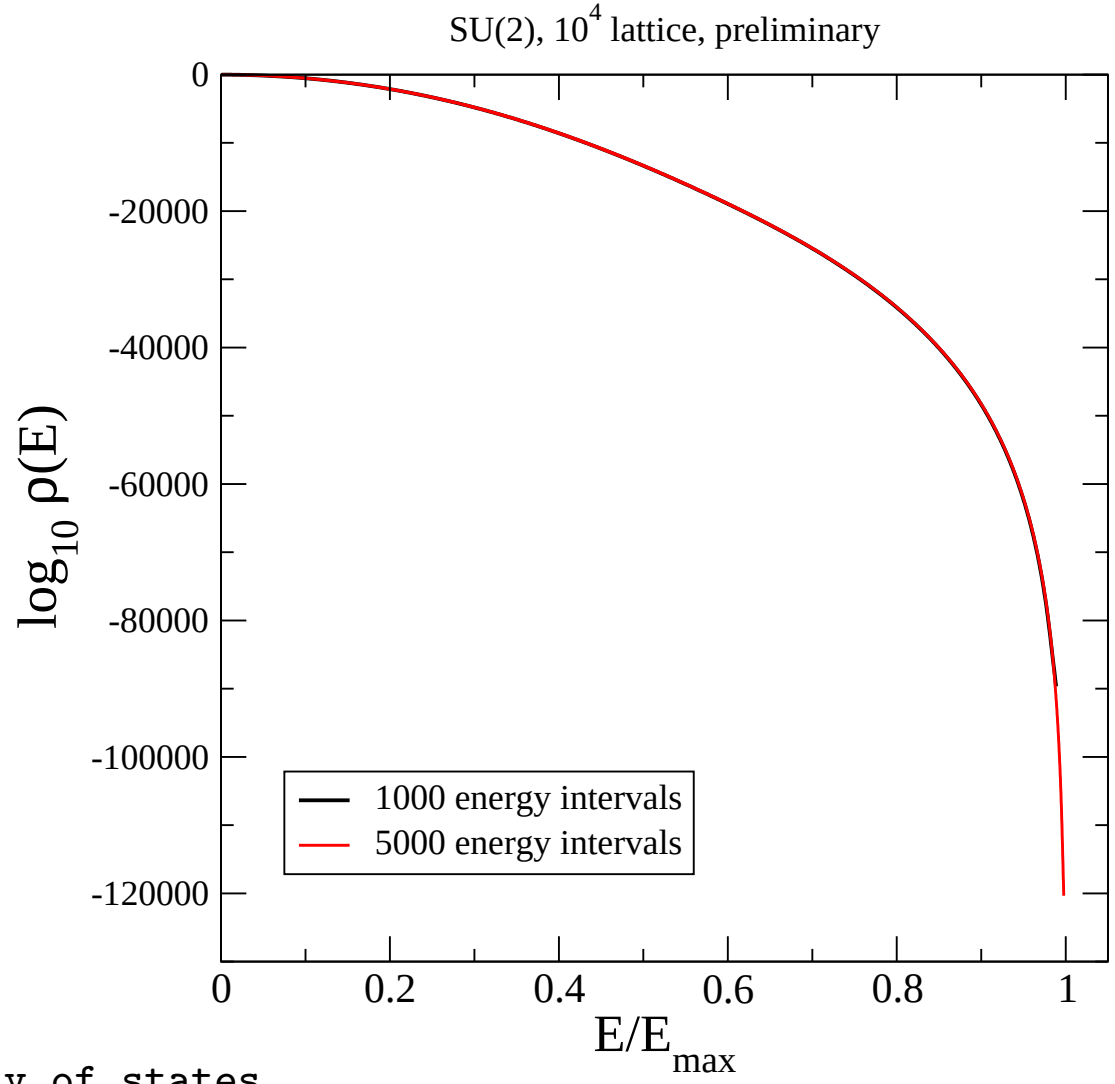
[Langfeld,Lucini,Rago, The density of states in gauge theories, arXiv:1204.3243 [hep-lat]]

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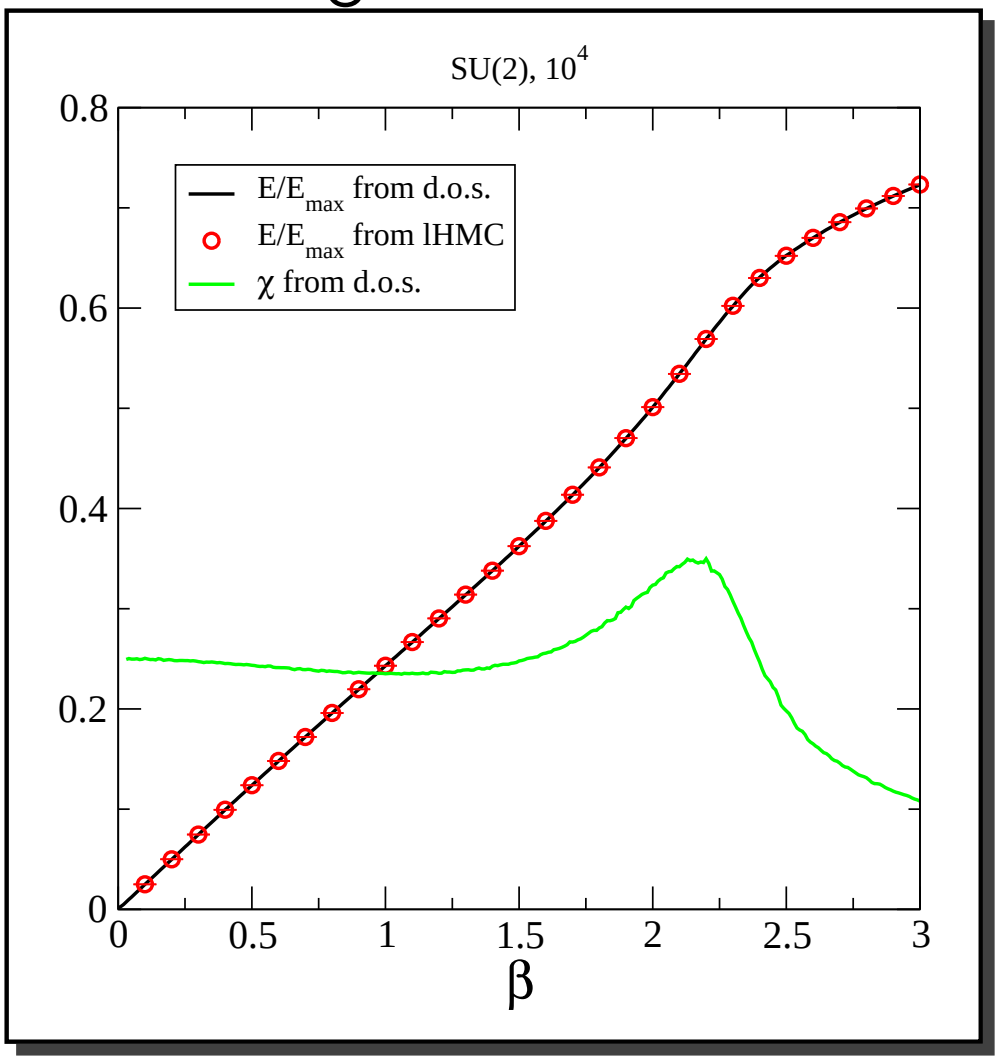
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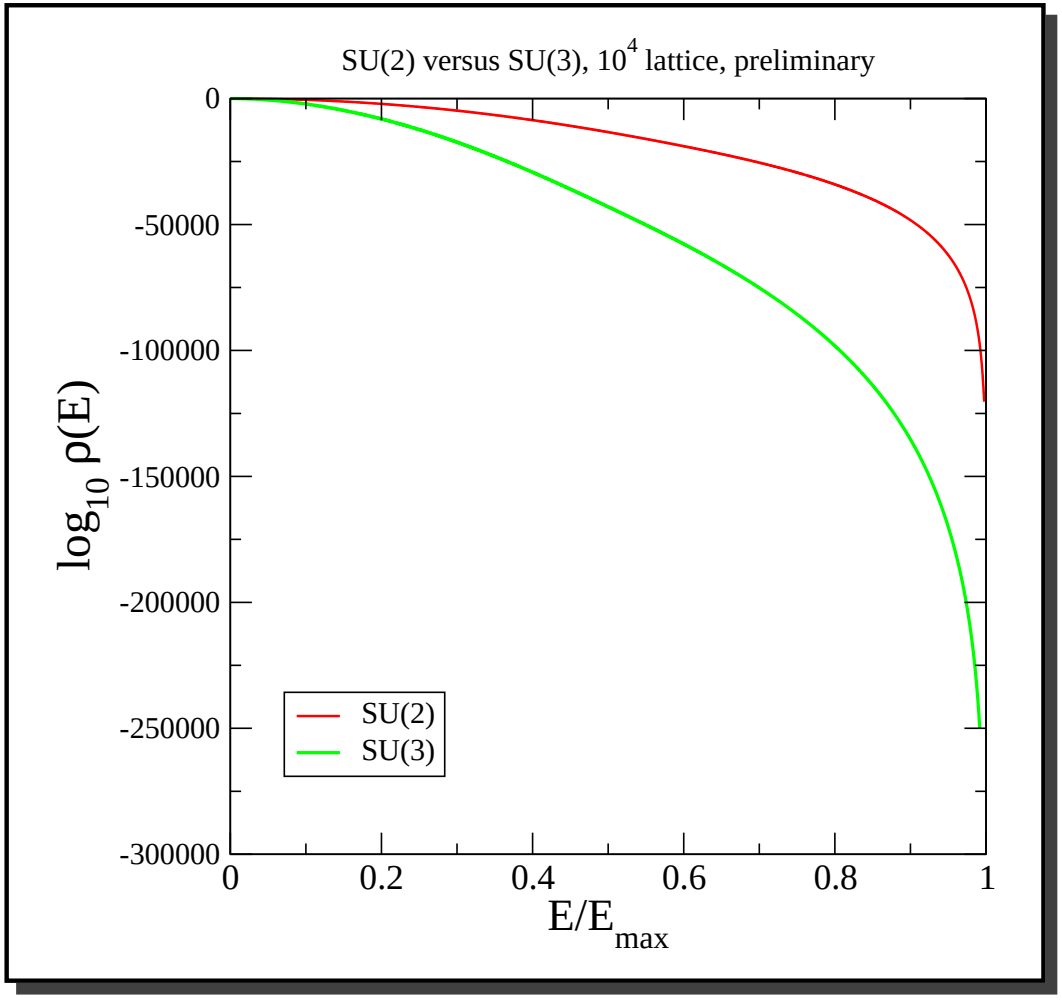
- calculating observables:

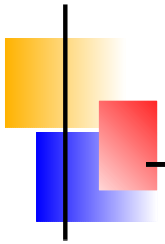


- $\langle E \rangle / V$
- $\chi = \langle E^2 \rangle - \langle E \rangle^2$

The density of states - SU(3)

- preliminary results for SU(3):

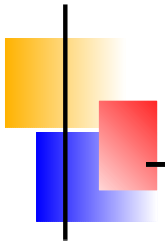




The density of states - conclusions

- more gauge theories to come: SO(3), SU(N),...
  more observables to calculate:

  `thermodynamics, Wilson loops` (E_0)`, fermion systems [GN model], etc...`



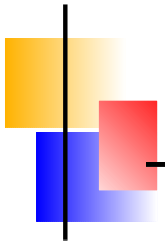
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- aim: study finite density systems

let's keep moving..

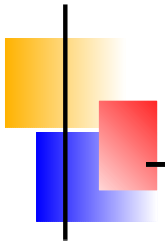


Outlook

- 2+1 O(2) $\Rightarrow$ well understood interacting QFT with access to finite densities!

  $\rightarrow$ ideal testbed for new methods

  e.g. [Aarts, James, JHEP 1008 (2010) 020]



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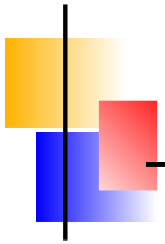
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- generalised density of states:

$$P_\beta(s) = \int \mathcal{D}\phi \; \delta\Big(s - S_I[\phi]\Big) \; \exp\{\beta S_R[\phi]\}$$

  S_I: imaginary part of the action

  S_R: real part of the action



Outlook

- Partition function: $$Z = \int ds\; P_\beta(s)\; \exp\{i\,\beta\,s\}$$
- Hope: $\log\, P_\beta(s)$ smooth function

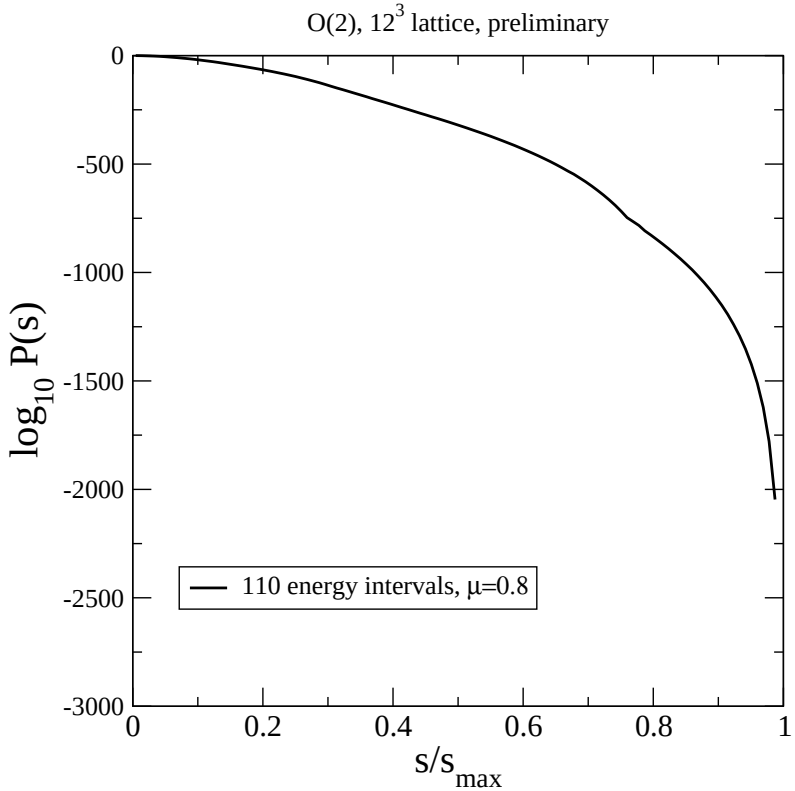
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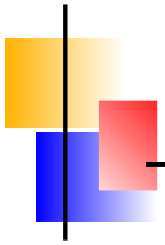
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- `Preliminary` result:



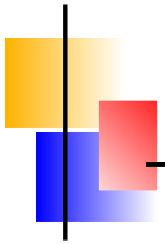
[KL, HPCC preliminary]



Conclusions

- recursive calculation of the density of states works brilliantly! *(more to come)*

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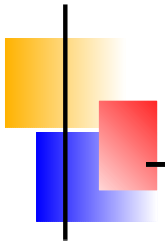
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- introduced the generalised density of states for complex action theories
  $P(s)$ smooth $\leftrightarrow$ oscillation $\propto V$

  `[results are upcoming]`

The density of states - compact U(1)

- studv phase transition in U(1):



△ weakly first order

[Arnold, Bunk, Lippert,
Schilling,
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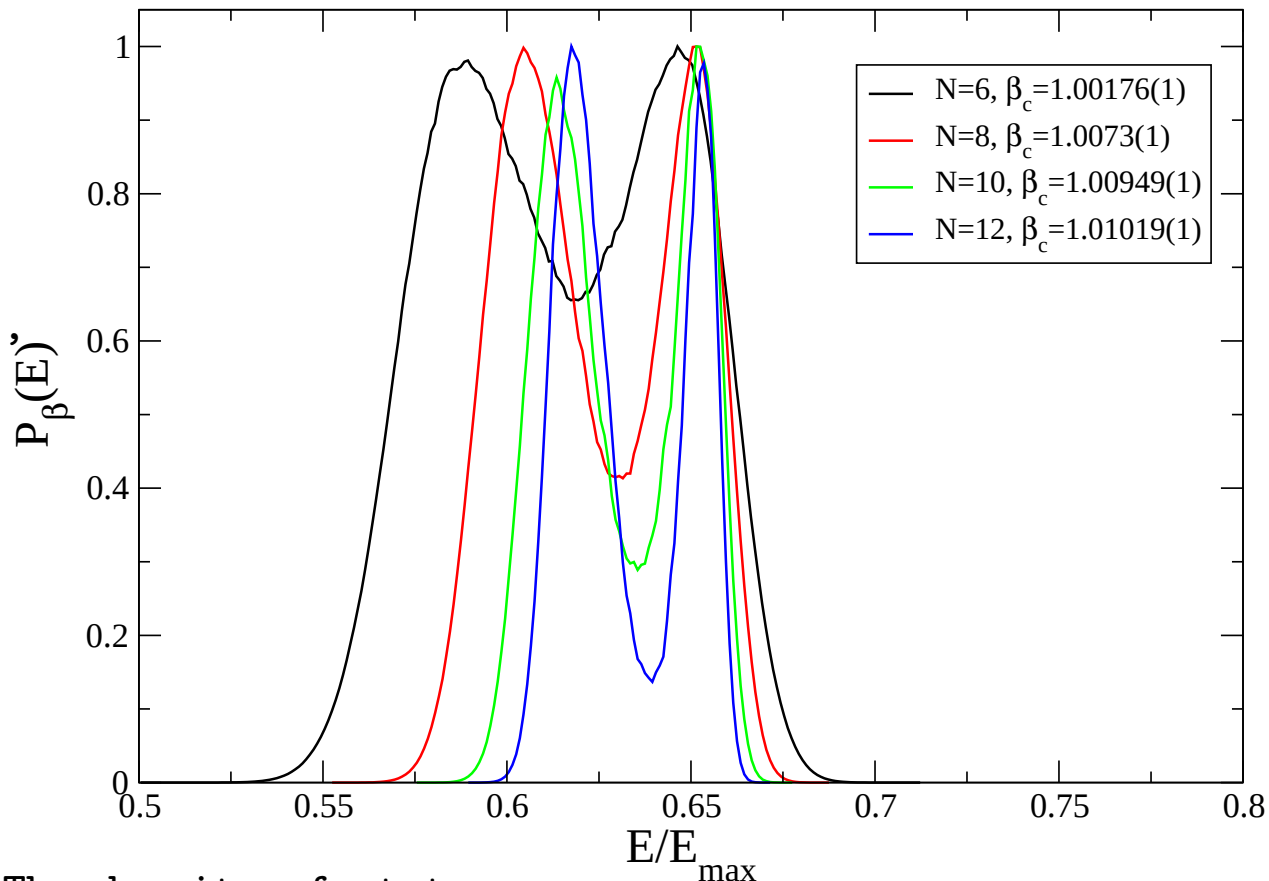
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