Towards hot and/or dense QCD from first principles

HIC for FAIR
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Owe Philipsen

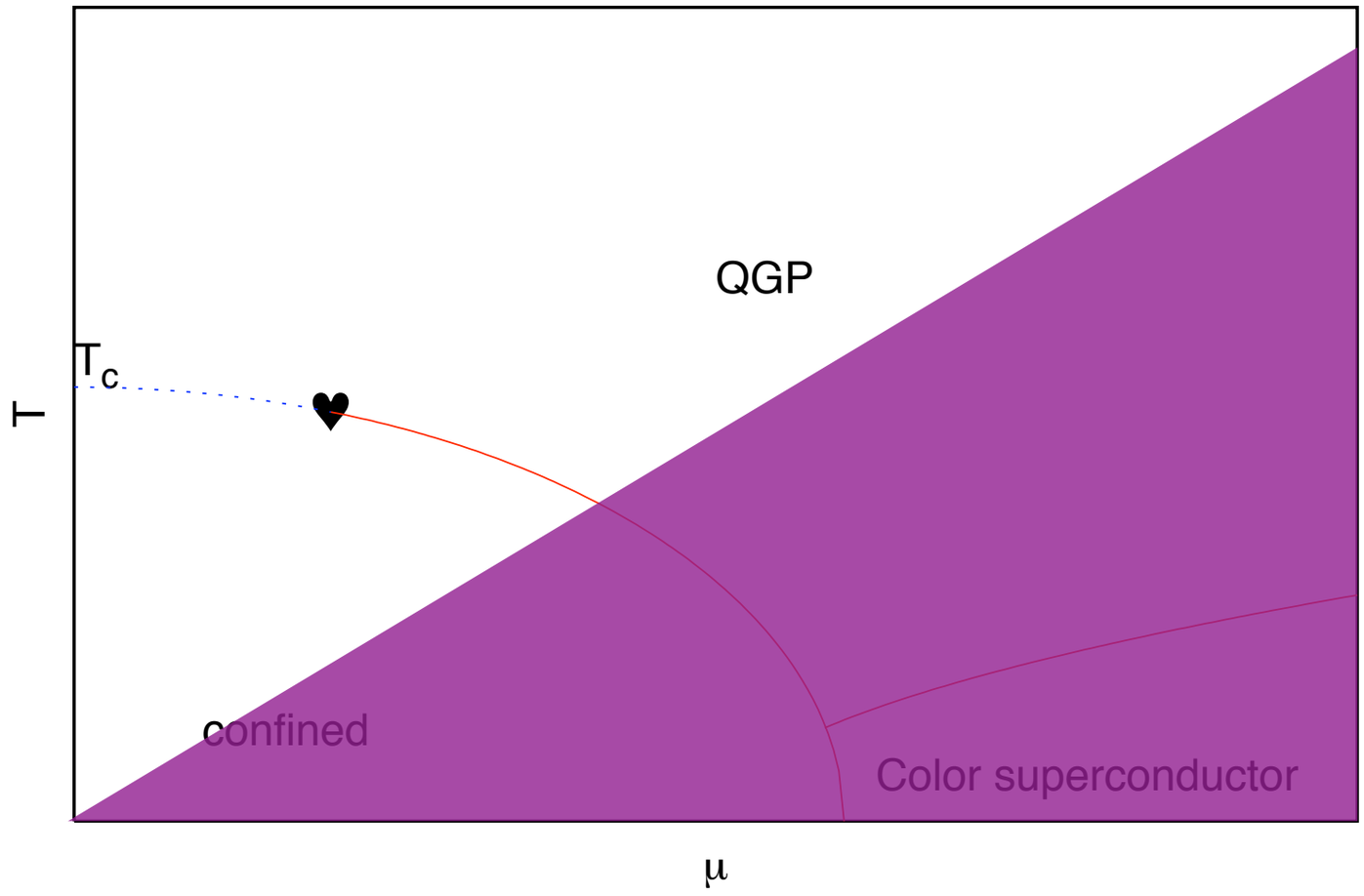
GOETHE UNIVERSITÄT FRANKFURT AM MAIN

in collaboration with M. Fromm, J. Langelage, S. Lottini, M. Neuman

- Introduction: The QCD phase diagram
- A 3d effective lattice theory derived by strong coupling methods
- The deconfinement transition in Yang-Mills theory
- The deconfinement transition in QCD with heavy dynamical quarks
- Cold and dense QCD

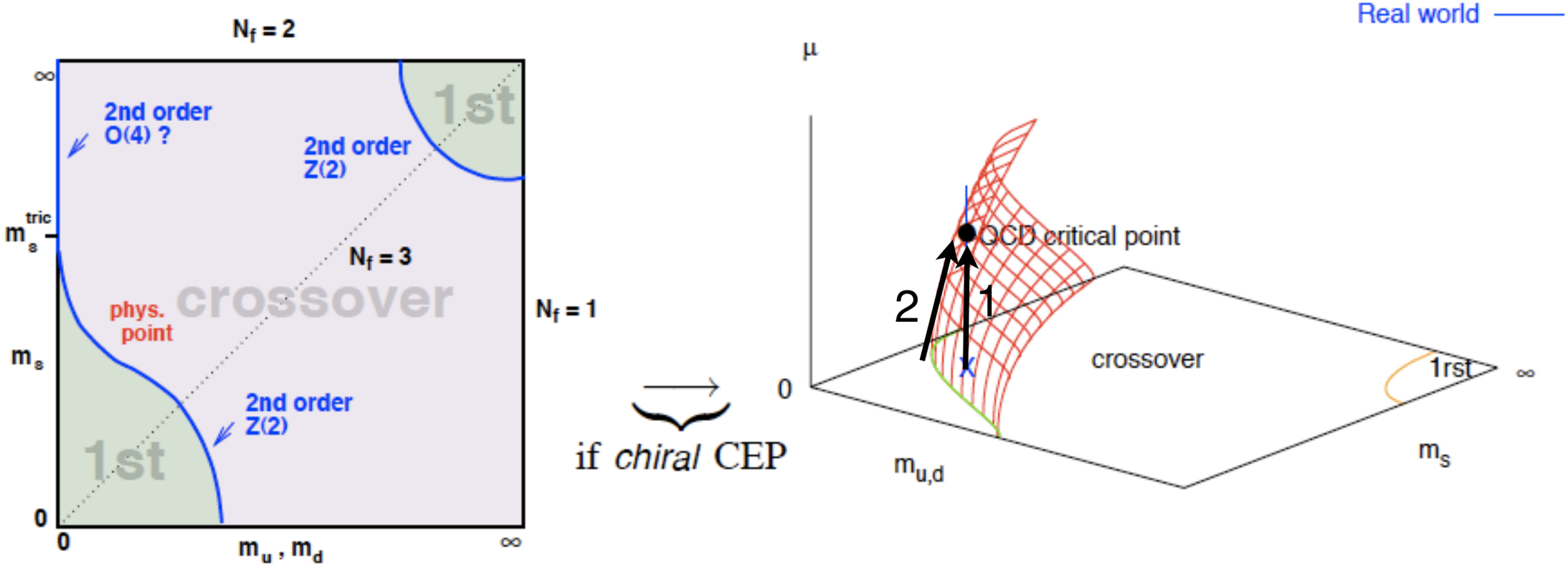
The (lattice) calculable region of the phase diagram

order of p.t.
at zero density
depends on Nf, quark mass



- Sign problem prohibits direct simulation, circumvented by approximate methods: reweigthing, Taylor expansion, imaginary chem. pot., need $\mu/T \lesssim 1 \;\; (\mu = \mu_B/3)$
- Upper region: equation of state, screening masses, quark number susceptibilities etc. under control
- Here: phase diagram itself, so far based on models, most difficult!

Is there a QCD critical point?



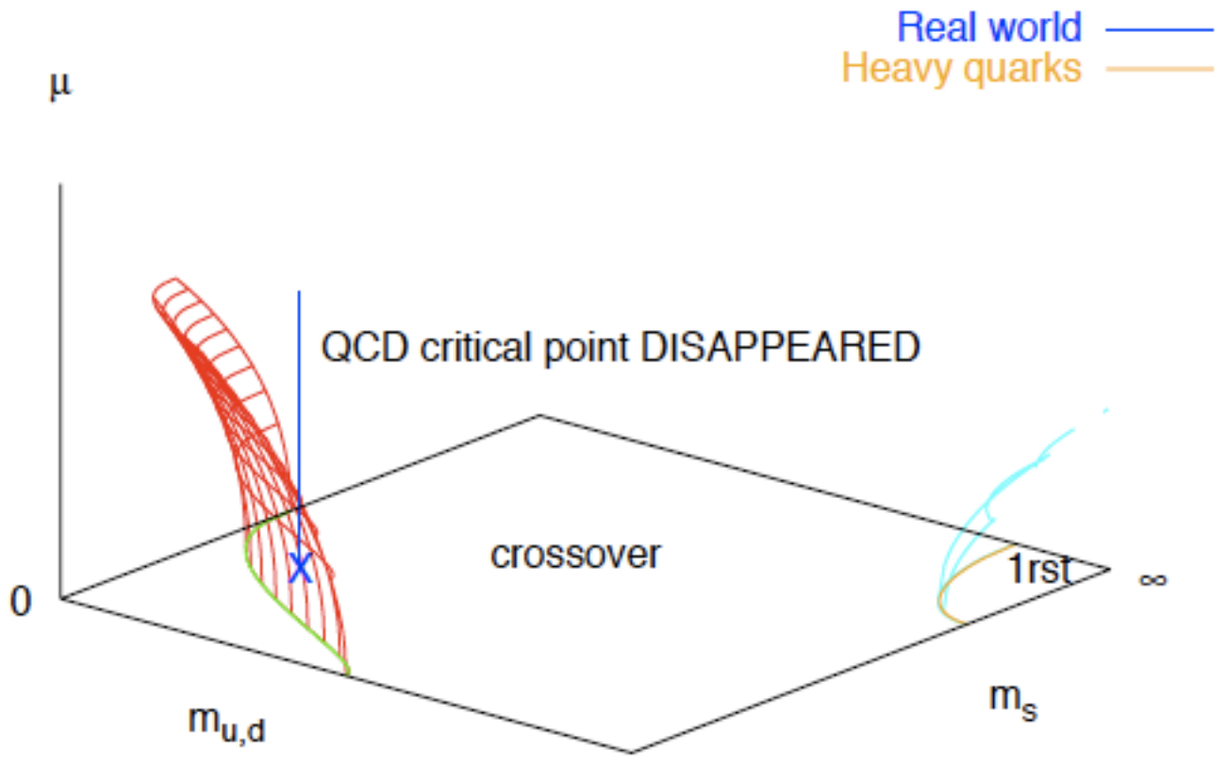
Two strategies:

1 follow vertical line: $m = m_{\text{phys}}$, turn on μ

2 follow critical surface: $m = m_{\text{crit}}(\mu)$

Some methods trying (1) give indications of critical point, but systematics not yet controlled

On coarse lattice exotic scenario: no chiral critical point at small density



de Forcrand, O.P. 08,09

Weakening of p.t. with chemical potential also for:

-Heavy quarks — Fromm, Langelage, Lottini, O.P. 11

-Light quarks with finite isospin density — Kogut, Sinclair 07

-Electroweak phase transition with finite lepton density — Gynther 03

Large densities? Try effective theories!

- E.w. phase transition: success with dimensional reduction!
- Scale "separation": $g^2 T < gT < 2\pi T$

  Integrate hard scale perturbatively, treat eff. 3d theory on lattice, valid for sufficiently weak coupling

- Does not work for QCD, perturbative dim. red. breaks Z(3) of YM theory
- Bottom-up construction of Z(N)-invariant theory by matching:

  works for SU(2), unfinished for SU(3)
  Vuorinen, Yaffe; de Forcrand, Kurkela; Kurkela, Vuorinen....

- Here: solution by strong coupling expansion!

Starting point: Wilson's lattice Yang-Mills action

Partition function; link variables as degrees of freedom

$$Z = \int \prod_{x,\mu} dU(x;\mu) \exp\left(-S_{YM}\right) \equiv \int DU \exp\left(-S_{YM}\right)$$

Wilson's gauge action

$$S_W = -\frac{\beta}{N} \sum \mathrm{ReTr}(U_p) = \sum S_p \qquad \beta = \frac{2N}{g^2}$$

Plaquette: $\square \rightarrow 1 + ia^2 g F_{\mu\nu} - \frac{a^4 g^2}{2} F_{\mu\nu} F^{\mu\nu} + O(a^6) + \ldots$

$U_\mu(x) = \mathrm{e}^{-\mathrm{i}agA_\mu(x)}$

$$T = \frac{1}{aN_t} \qquad \text{continuum limit} \quad a \to 0, N_t \to \infty$$

Small $\beta(a) \Rightarrow$ small T

The effective theory, Yang-Mills

- Split temporal and spatial link integration and use character expansion ($a_r(\beta)$: expansion parameter of representation r)

$$Z = \int [dW] \exp \left\{ \ln \int [dU_i] \prod_p \left[1 + \sum_{r \neq 0} d_r a_r(\beta) \chi_r(U_p) \right] \right\}$$

$$\equiv \int [dW] \exp\left[-S_{eff}\right] \qquad W(\vec{x}) = \prod_{\tau=1}^{N_\tau} U_0(\tau, \vec{x})$$

Expansion parameter: $u = a_f(\beta) = \beta/18 + \cdots$

$$-S_{eff} = \lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 S_3 + \ldots$$

- S_n depend only on Polyakov loops

- Leading order graph in case of $N_\tau = 4$:

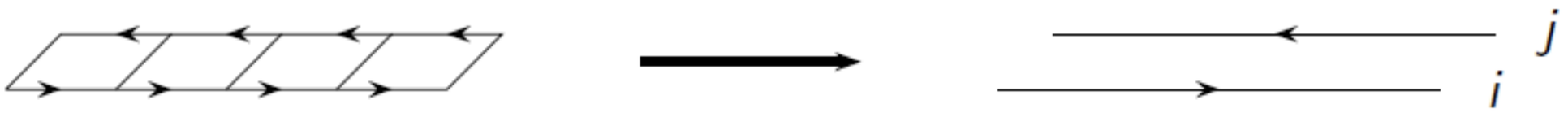


Figure: 4 plaquettes in fundamental representation lead to a 2 Polyakov loop interaction term

- Integration of spatial link variables leads to

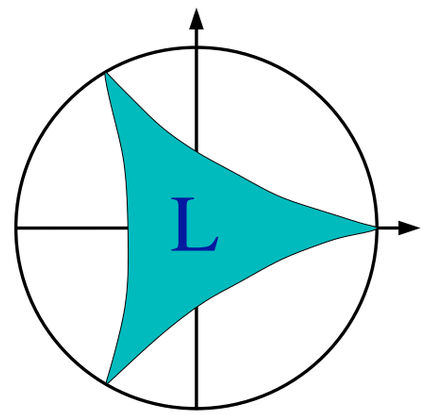
$$-S_1 = u^{N_\tau} \sum_{\langle ij \rangle} \mathrm{tr}\, W_i \,\mathrm{tr}\, W_j$$

- Possible generalizations: larger distance, higher dimensional representations, larger number of loops involved, ...

Details: poster Jens Langelage!

Effective one-coupling theory for SU(3) YM

$$
\begin{aligned}
Z &= \int [dL] \exp\left[-S_1 + V_{SU(3)}\right] \\
&= \int [dL] \prod_{\langle ij \rangle} \left[1 + 2\lambda_1 \mathrm{Re}\left(L_i L_j^*\right)\right] * \\
&\quad * \prod_i \sqrt{27 - 18|L_i|^2 + 8\mathrm{Re}L_i^3 - |L_i|^4}
\end{aligned}
$$



$$
\lambda(u, N_\tau \geq 5) = u^{N_\tau} \exp\left[N_\tau \left(4u^4 + 12u^5 - 14u^6 - 36u^7 + \frac{295}{2}u^8 + \frac{1851}{10}u^9 + \frac{1055797}{5120}u^{10}\right)\right]
$$

Subleading couplings

Subleading contributions for next-to-nearest neighbours:

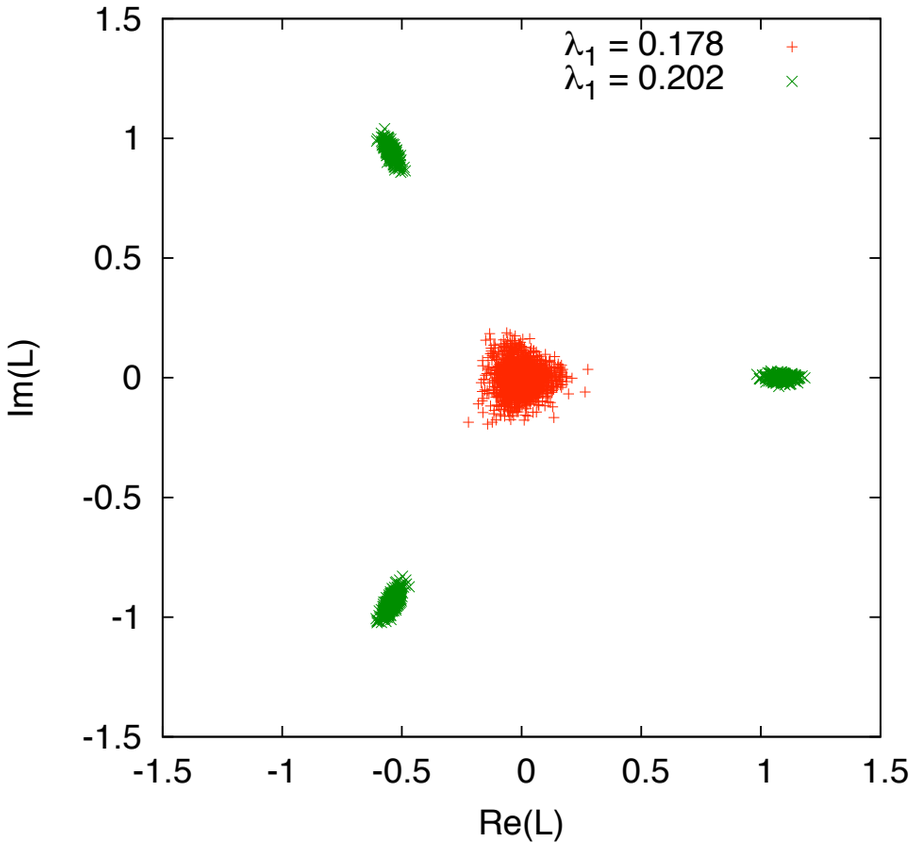
$$\lambda_2 S_2 \quad \propto \quad u^{2N_\tau+2} \sum_{[kl]}{}' 2\mathrm{Re}(L_k L_l^*) \quad \text{distance} \; = \sqrt{2}$$

$$\lambda_3 S_3 \quad \propto \quad u^{2N_\tau+6} \sum_{\{mn\}}{}'' 2\mathrm{Re}(L_m L_n^*) \quad \text{distance} \; = 2$$

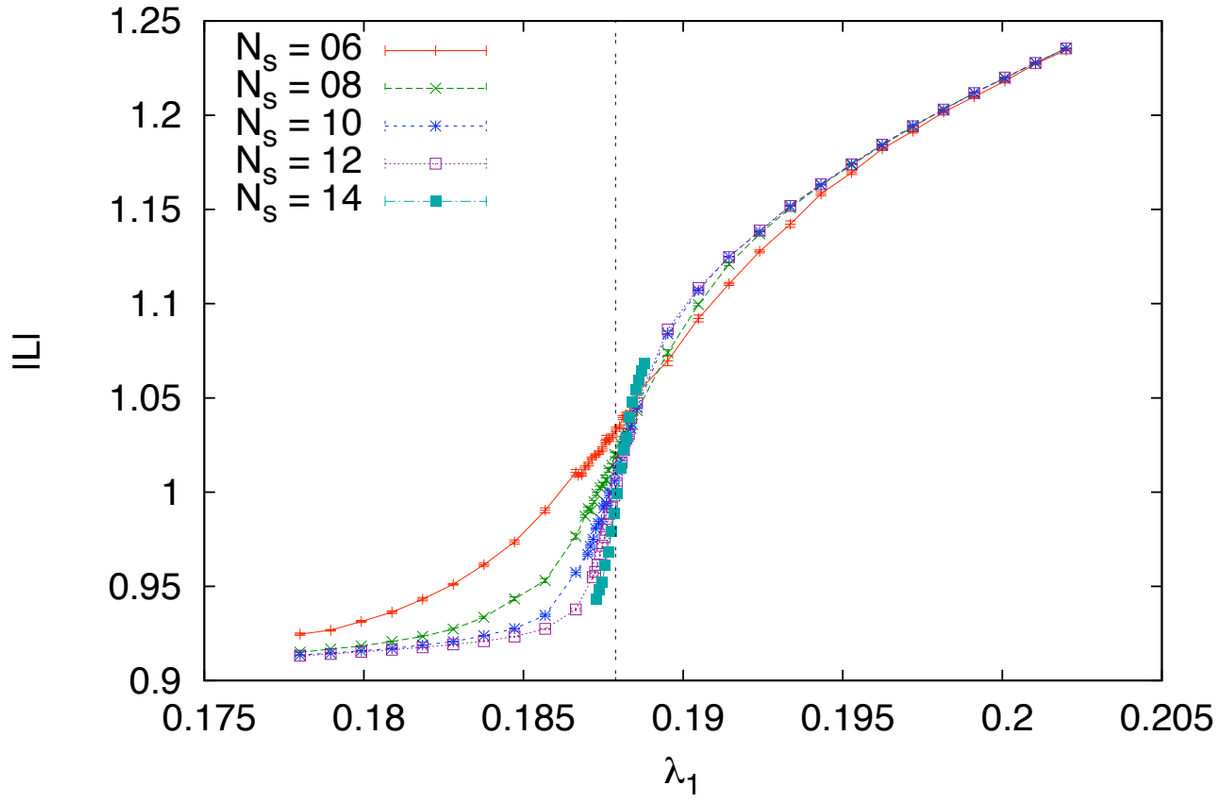
as well as terms from loops in the *adjoint* representation:

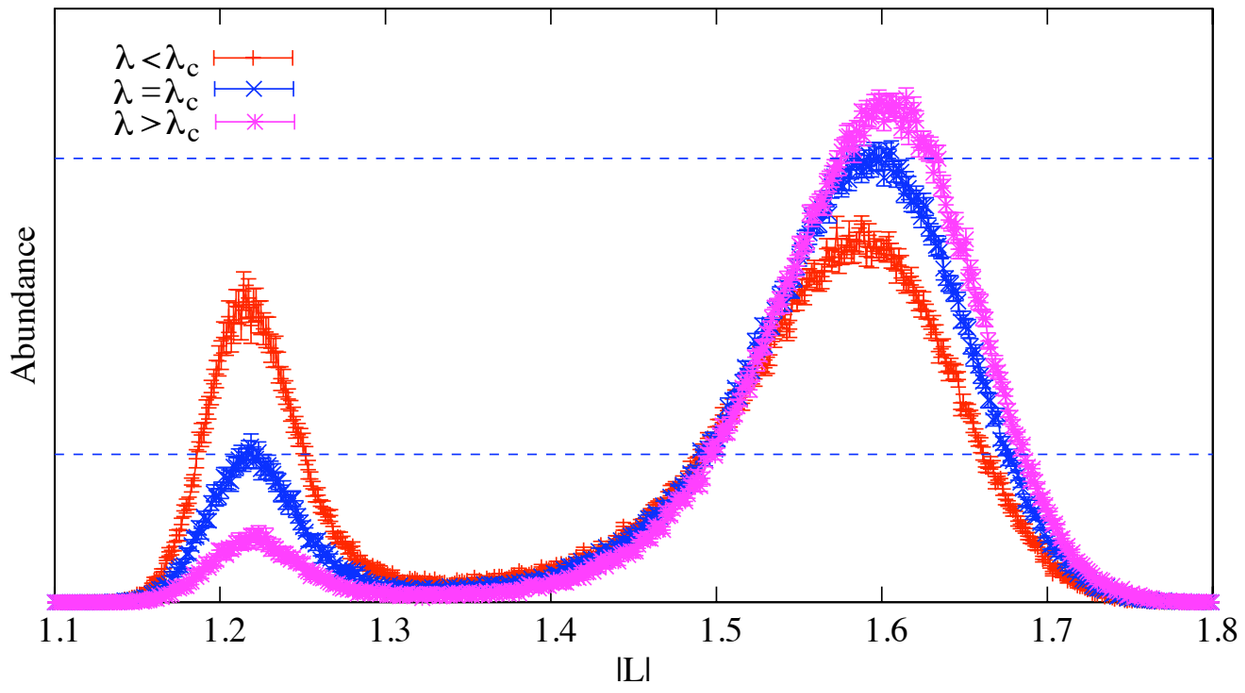
$$\lambda_a S_a \propto u^{2N_\tau} \sum_{<ij>} \mathrm{Tr}^{(a)} W_i \mathrm{Tr}^{(a)} W_j \quad ; \quad \mathrm{Tr}^{(a)} W = |L|^2 - 1$$

Numerical results for SU(3)



Order-disorder transition

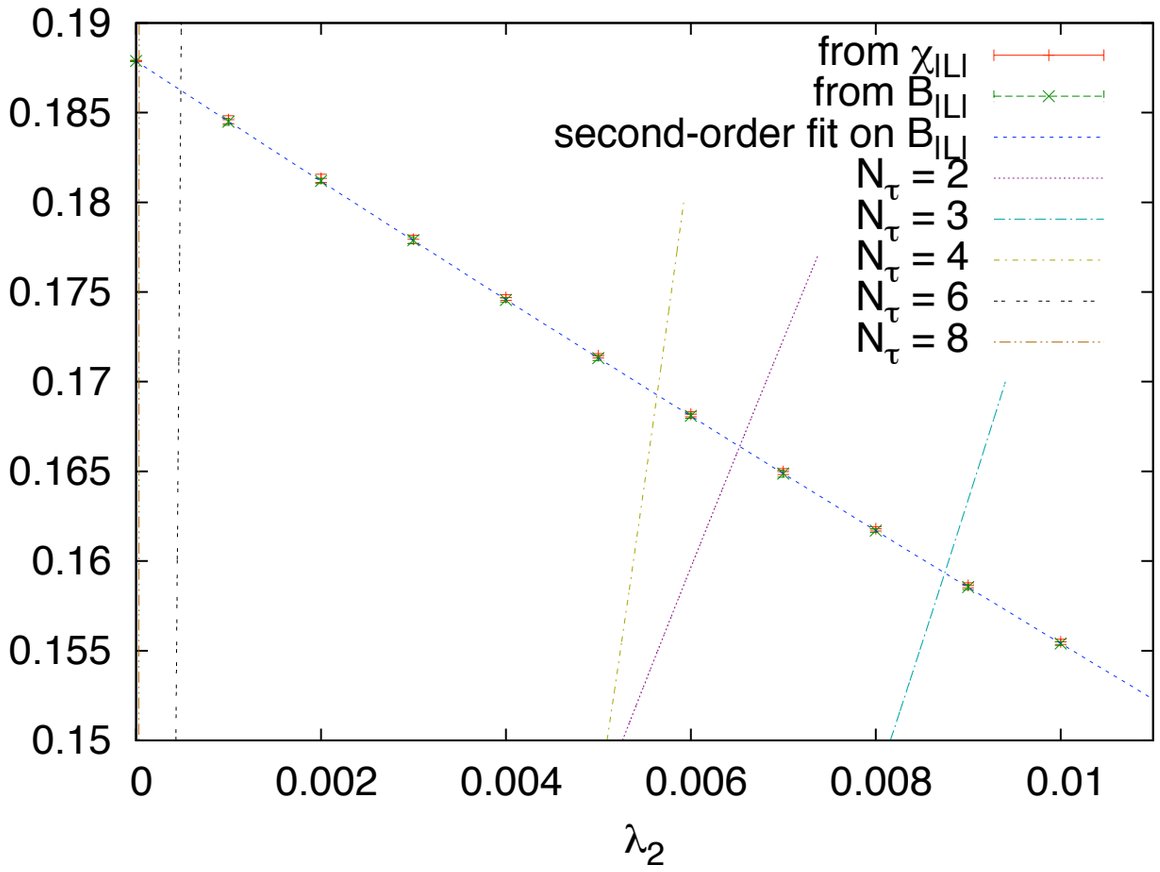


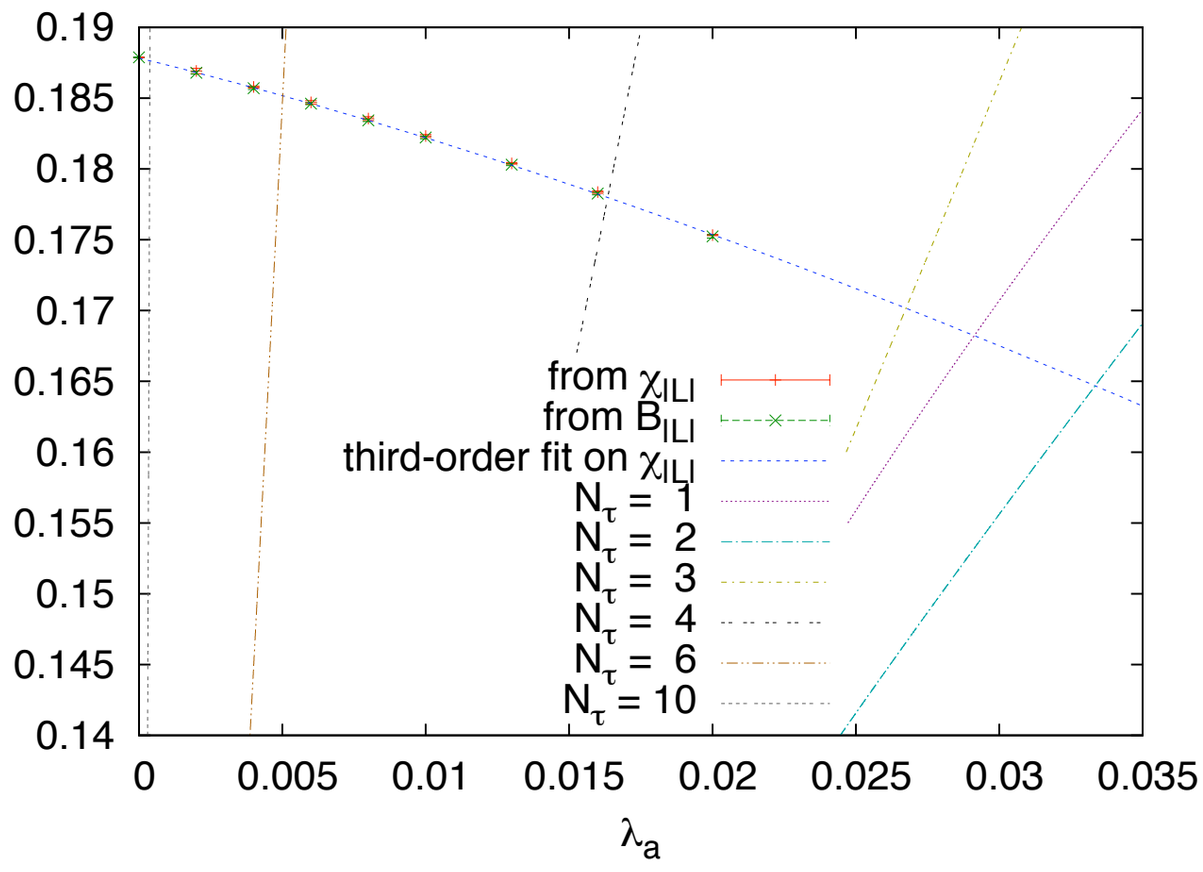


First order phase transition for SU(3) in the thermodynamic limit!

The influence of a second coupling

NLO-couplings: next-to-nearest neighbour, adjoint rep. loops





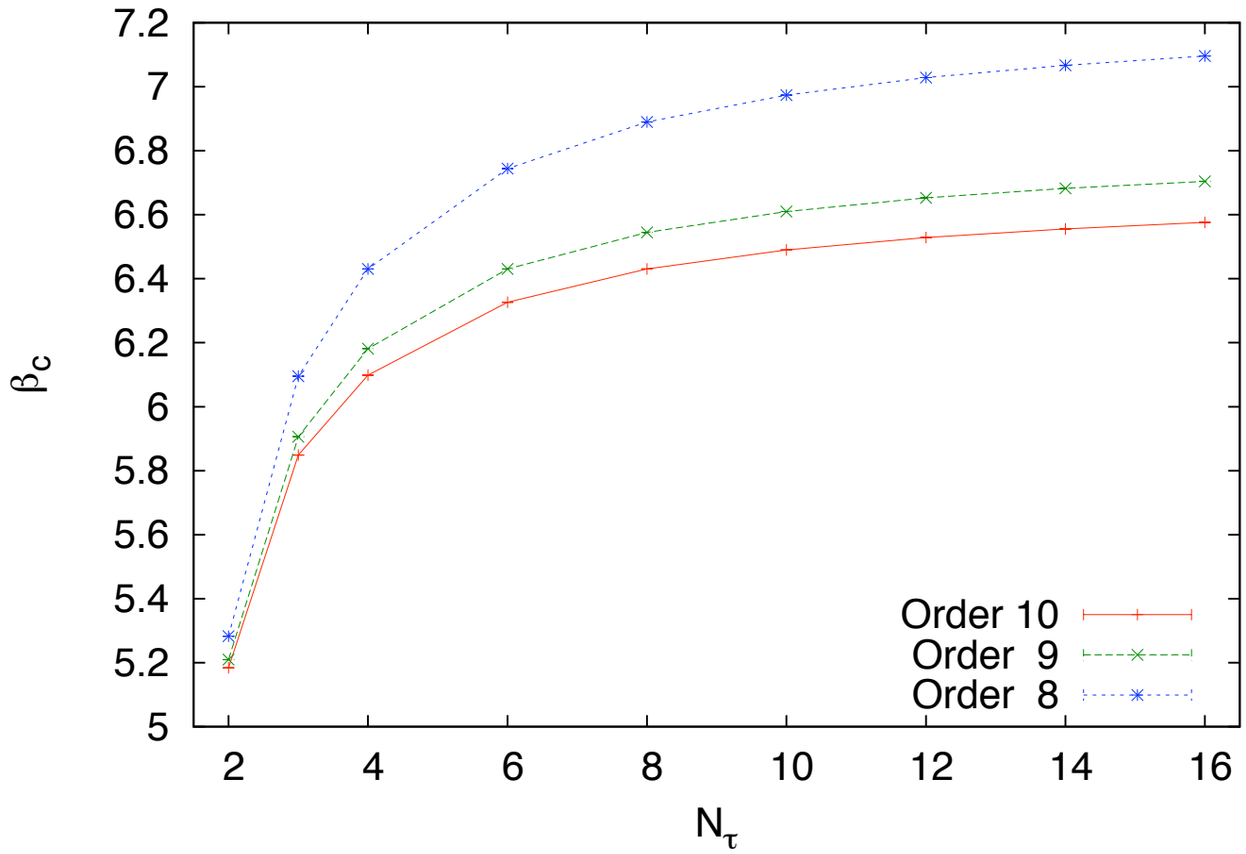
...gets very small for large N_τ !

Mapping back to 4d finite T Yang-Mills

Inverting

$$\lambda_1(N_\tau, \beta) \rightarrow \beta_c(\lambda_{1,c}, N_\tau)$$

...points at reasonable convergence

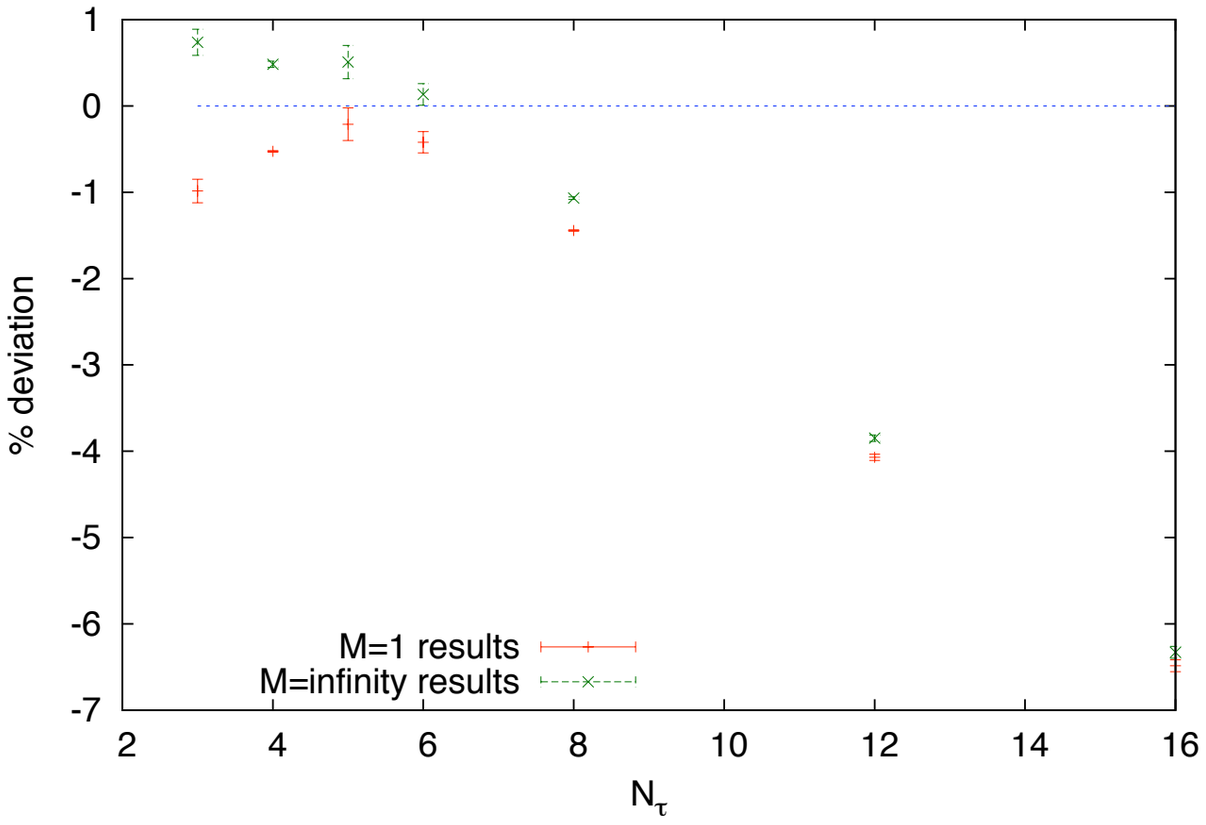


SU(3)

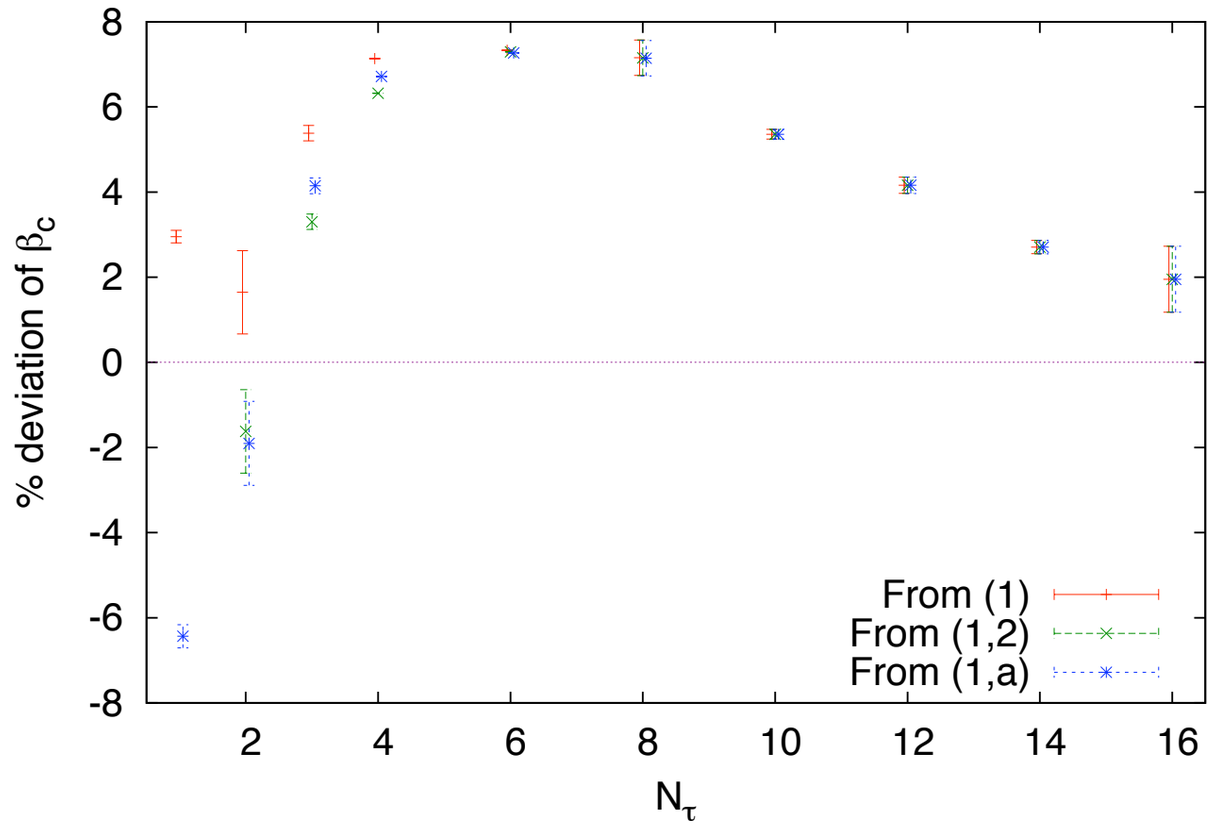
Comparison with 4d Monte Carlo

Relative accuracy for β_c compared to the full theory

SU(2)

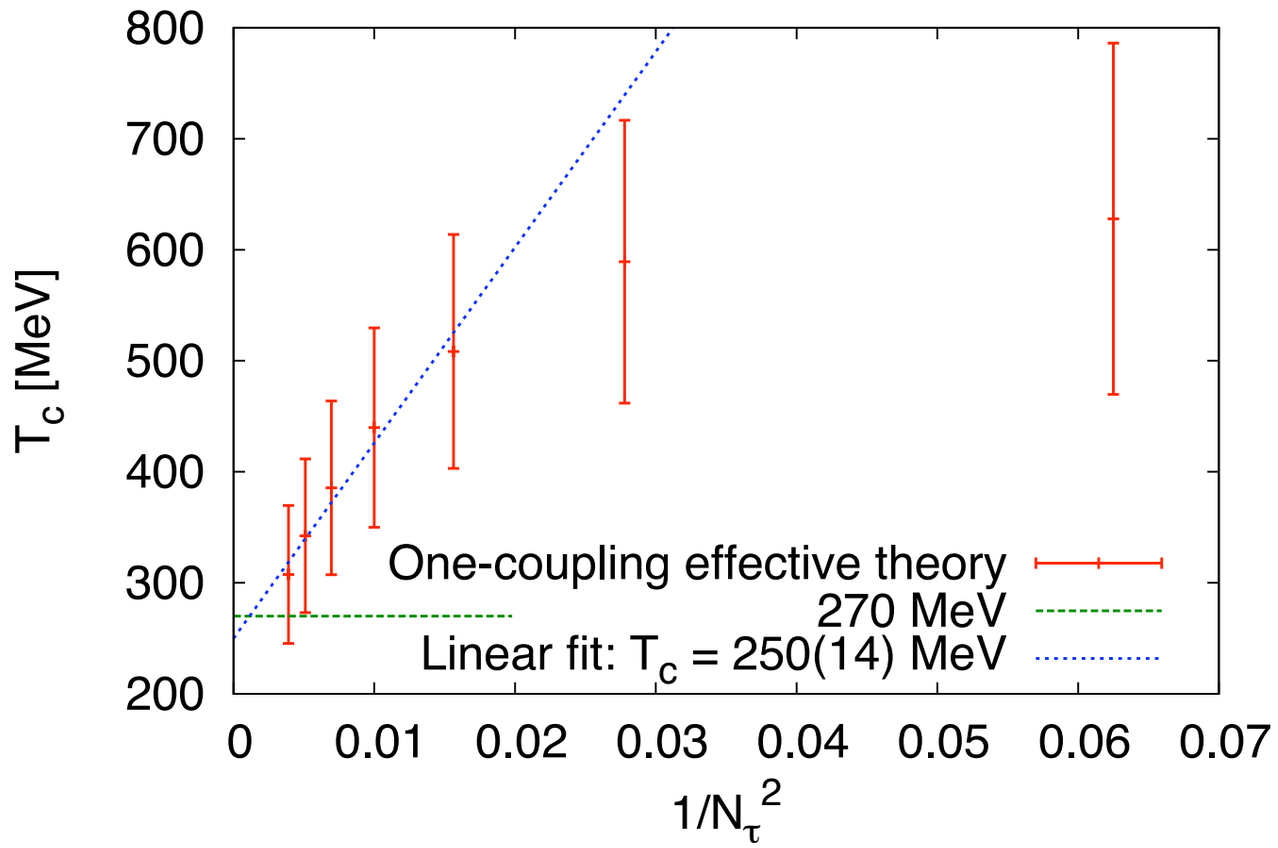


SU(3)



Note: influence of additional couplings checked explicitly!

Continuum limit feasible!



-error bars: difference between last two orders in strong coupling exp.

-using non-perturbative beta-function (4d T=0 lattice)

-all data points from one single 3d MC simulation!

Including heavy, dynamical Wilson fermions

N_f (degenerate) fermions $\Longrightarrow$ $S = S_{\text{gauge}} + S_q[U, \psi, \overline{\psi}]$

$$S_q = \sum_{x,y;f} \overline{\psi}_{f,y} \Big(\mathbb{1} - \kappa H[U] \Big)_{yx} \psi_{f,x} \quad , \quad H[U]_{yx} = \sum_{\pm\mu} \delta_{y,x+\hat{\mu}} (\mathbb{1} + \gamma_\mu) U_{x,\mu}$$

Integrate the Grassmann variables $\psi, \overline{\psi}$:

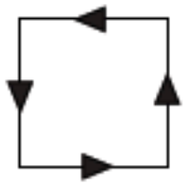
$$S = S_{\text{gauge}} - N_f \,\text{Tr}\log(\mathbb{1} - \kappa H)$$

Expand in the *hopping parameter* $\kappa = 1/(2aM + 8)$: [*]

$$S = S_{\text{gauge}} + N_f \sum_{\ell=1}^{\infty} \frac{\kappa^\ell}{\ell} \text{Tr} H[U]^\ell$$

Similar to de Pietri, Feo, Seiler, Stamatescu 07, Aarts, Stamatescu 08 ...

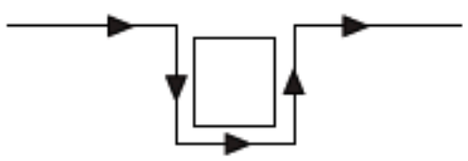
Links along imaginary time gain $\exp(\pm\mu a)$



reabsorbed in gauge part: $\begin{cases} \beta \to \beta + \mathcal{O}(\kappa^4) \\ u(\beta) \to u(\beta, \kappa) \end{cases}$

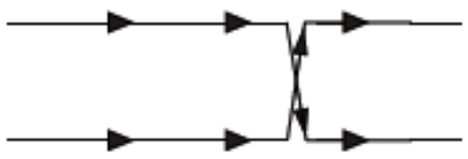


LO Polyakov "magnetic" term $\sim$ $\begin{cases} \underbrace{(2\kappa e^{+a\mu})^{N_\tau}}_{h_1} L \\ \underbrace{(2\kappa e^{-a\mu})^{N_\tau}}_{\overline{h}_1} L^* \end{cases}$



higher corrections to the above:

$$h_1 = (2\kappa e^{a\mu})^{N_\tau} \left[1 + \mathcal{O}(k^2) f(u) + \ldots\right]$$



other (suppressed) terms, such as $h_2(L_x L_{x+\hat{i}})$,

$$h_2 \sim (2\kappa e^{a\mu})^{2N_\tau} \kappa^2 \ldots$$

In general the model becomes (with $\overline{h}_i(\mu) = h_i(-\mu)$)

$$-S_{\rm eff} = \sum_i \lambda_i(u,\kappa,N_\tau) S_i^S - 2N_f \sum_i \left[h_i(u,\kappa,\mu,N_\tau) S_i^{\rm A} + \overline{h}_i(u,\kappa,\mu,N_\tau) S_i^{\dagger {\rm A}} \right]$$

Now, keep only $\lambda_1 S_1^S$ and $h_1 S_1^{\rm A} + \overline{h}_1 S_1^{\dagger {\rm A}}$ (now called just λ, h)

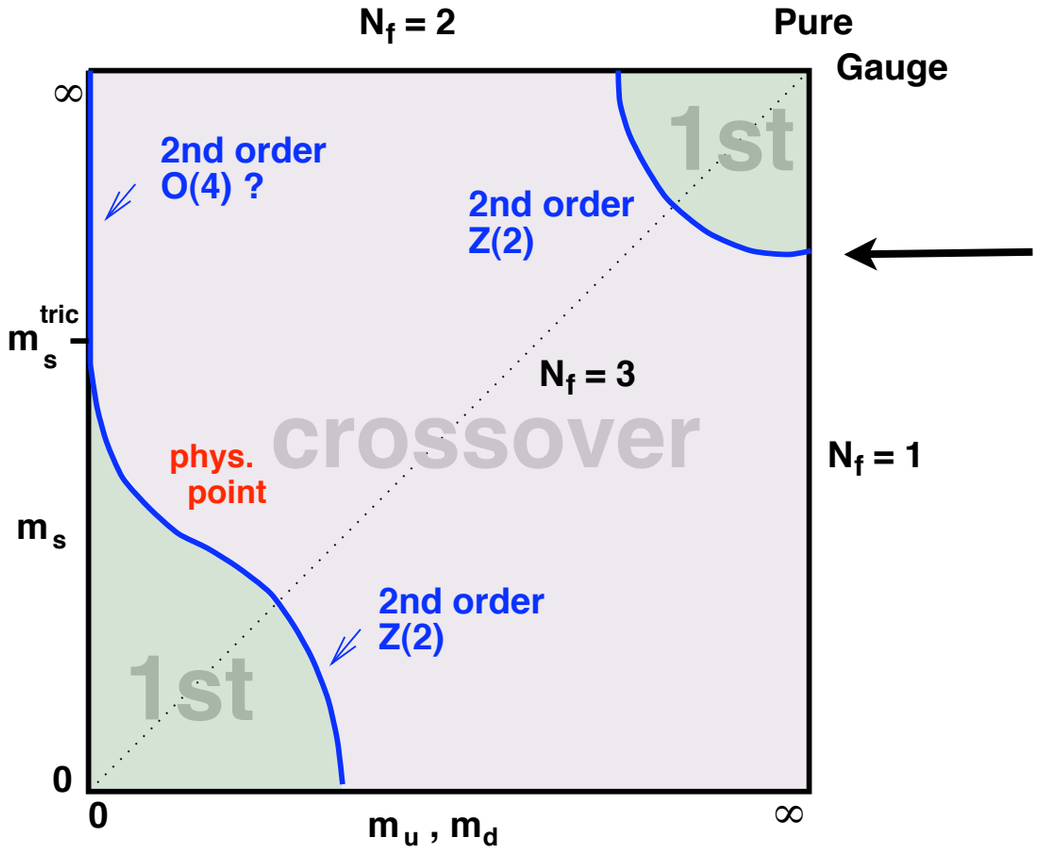
Higher powers of loops are resummed into a determinant:

$$Z_{\rm eff}(\lambda_1, h_1, \overline{h}_1; N_\tau) = \int [{\rm d}L] \Big(\prod_{<ij>} [1 + 2\lambda_1 {\rm Re} L_i L_j^*] \Big)$$
$$\Big(\prod_x \underbrace{\det[(1 + h_1 W_x)(1 + \overline{h}_1 W_x^\dagger)]^{2N_f}}_{\equiv Q(L_x, L_x^*)^{N_f}} \Big)$$

No chemical potential ($h = \overline{h}$): the full (λ, h)-model has then

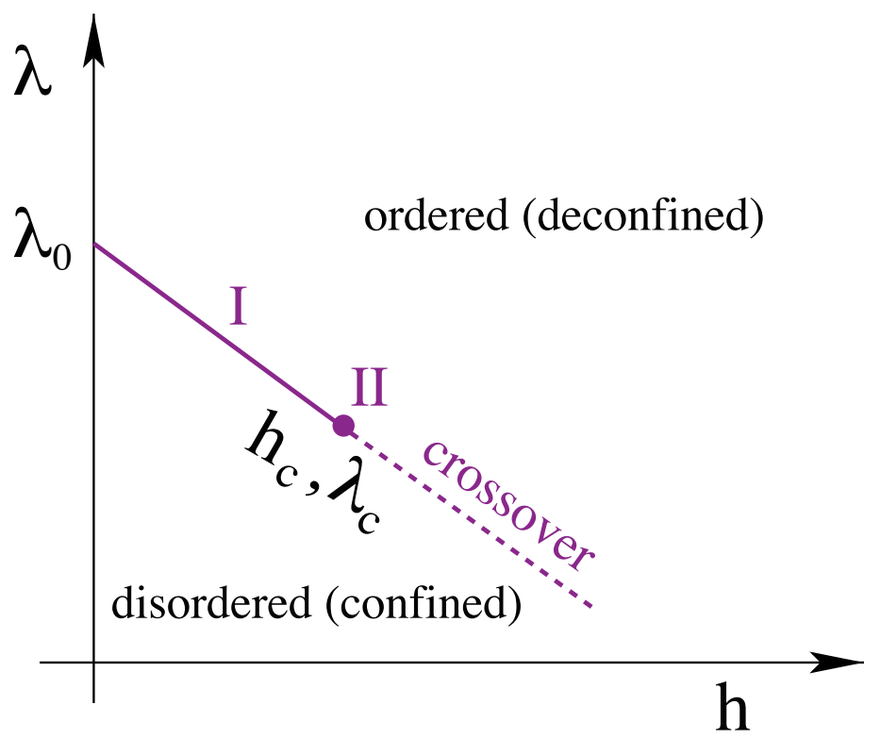
- a "spin-spin" interaction between neighbour Polyakov loops
- a "magnetic-field" term acting on sites

QCD: first order deconfinement transition region



deconfinement p.t.:
breaking of global $Z(3)$ symmetry;
explicitly broken by quark masses
transition weakens

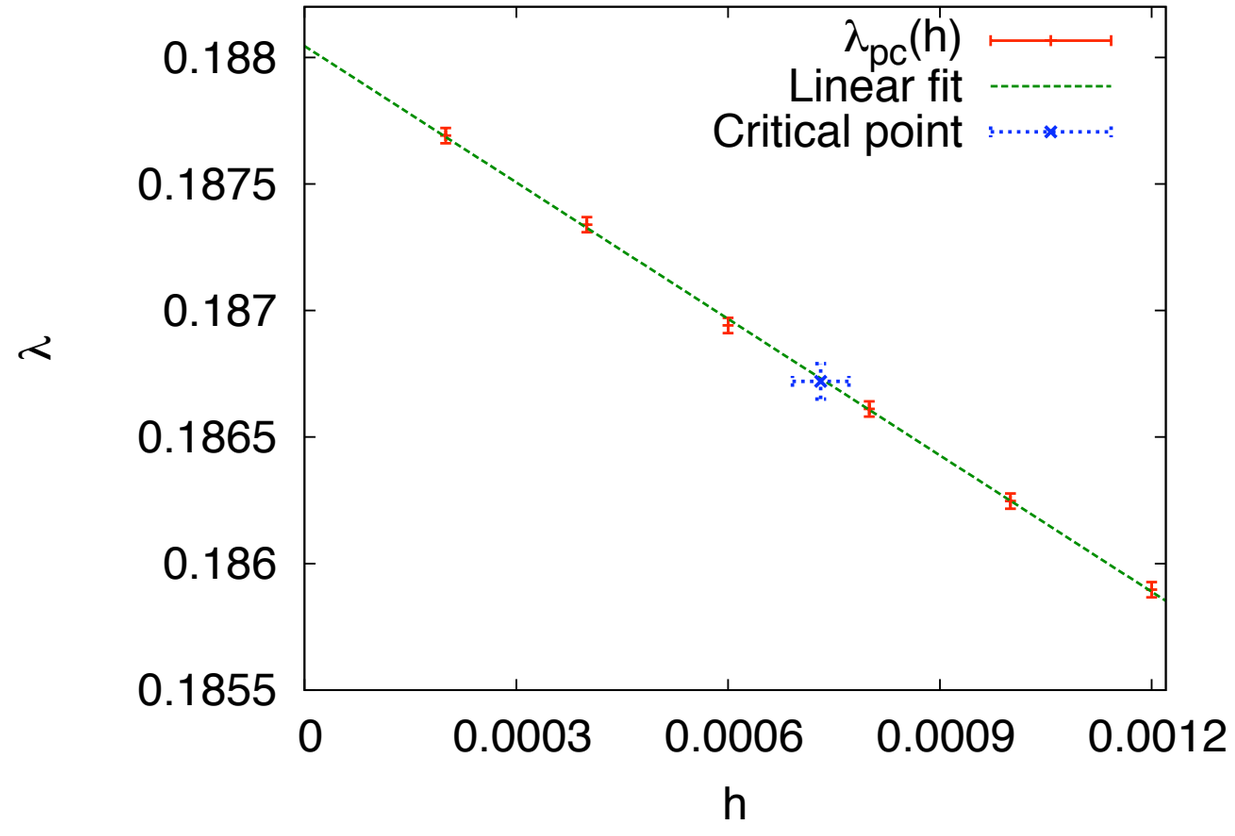
Phase diagram in eff. theory:



Phase boundary, numerically

To find $\lambda_{pc}(h)$, λ-scans were performed at various fixed h

- peak in $\chi_O = \langle O^2 \rangle - \langle O \rangle^2$
- dip in $B_O = 1 - \frac{\langle O^4 \rangle}{3\langle O^2 \rangle^2}$

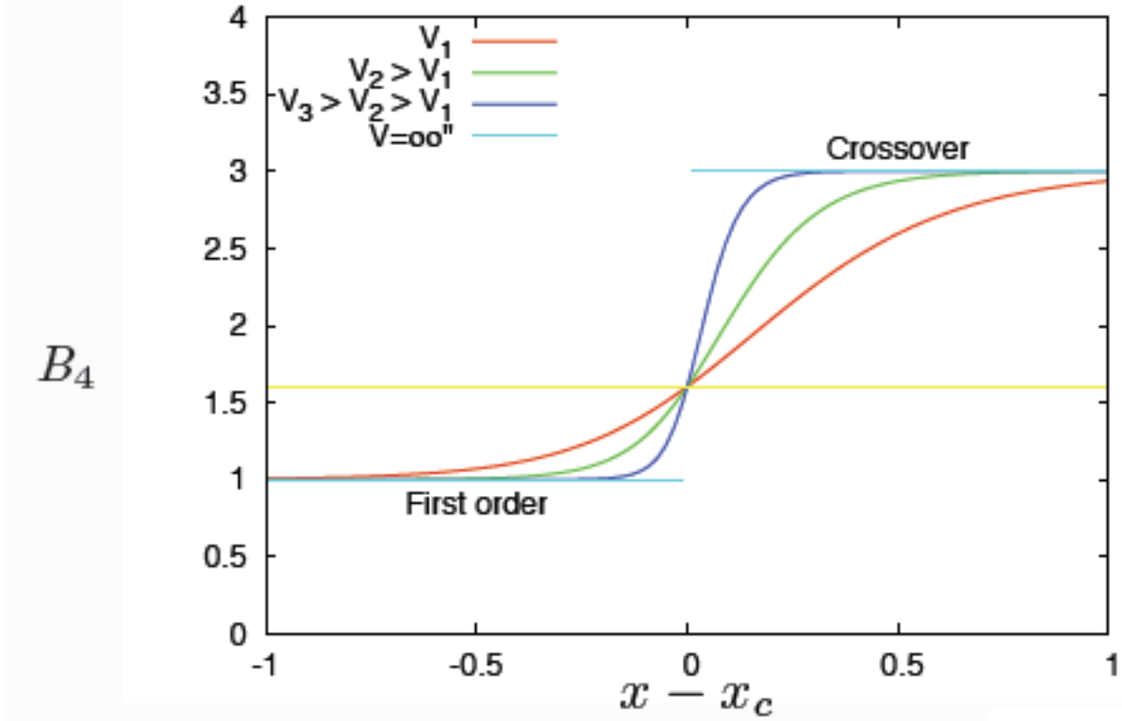


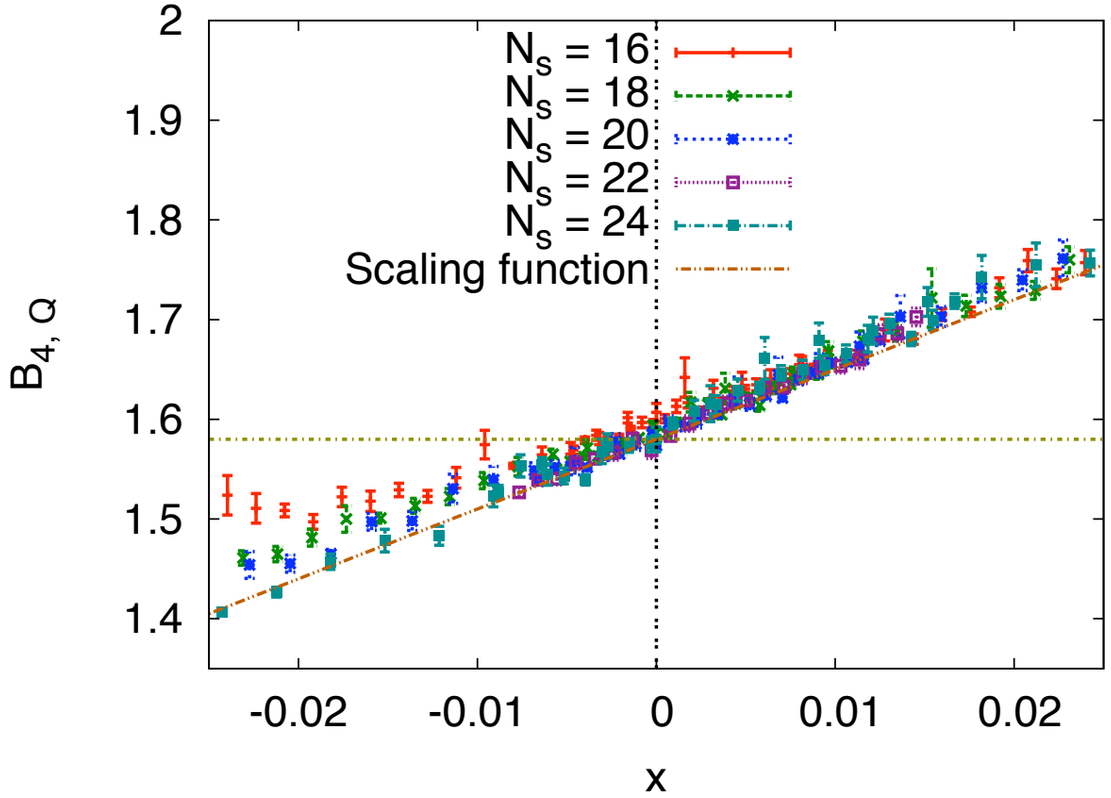
$$\lambda_{pc}(h) = 0.18805 - 1.797 \cdot h$$

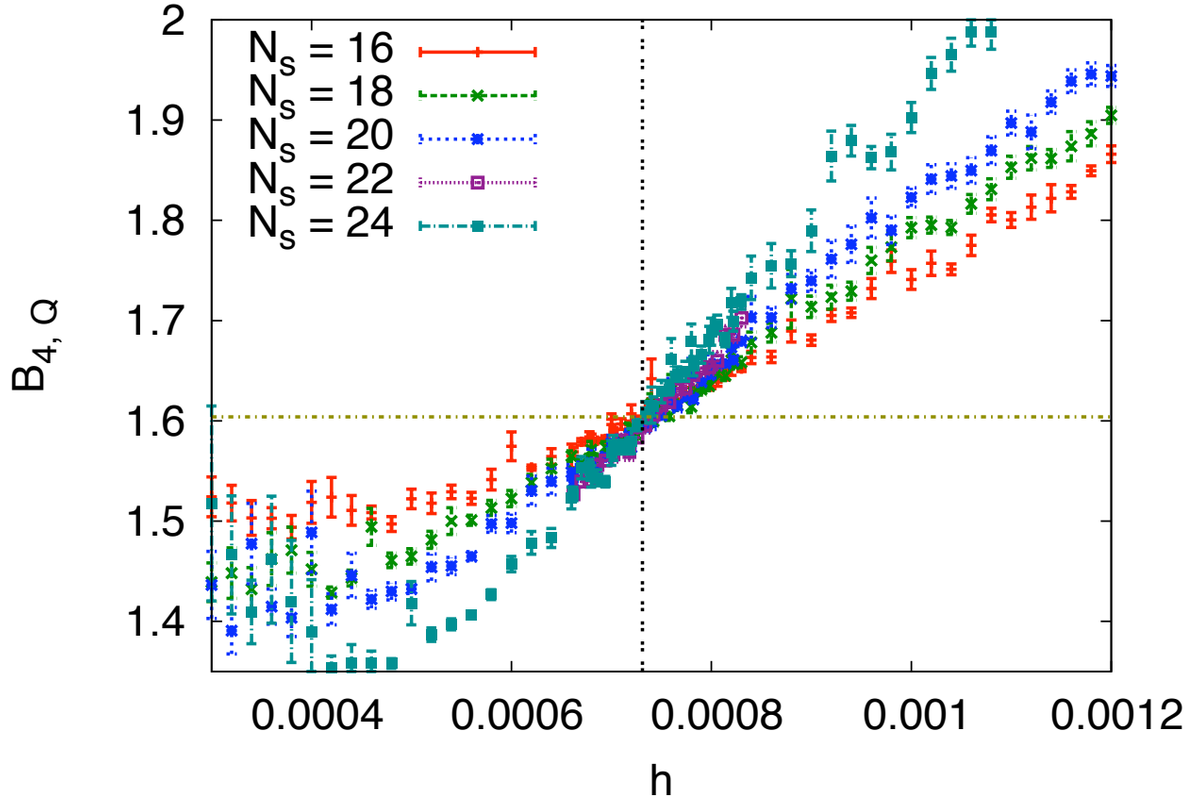
Observable to identify order of p.t.:

$$\delta B_Q = B_4(\delta Q) = \frac{\langle (\delta Q)^4 \rangle}{\langle (\delta Q)^2 \rangle^2}$$

$$B_4(x) = 1.604 + bL^{1/\nu}(x - x_c) + \ldots$$







The critical point

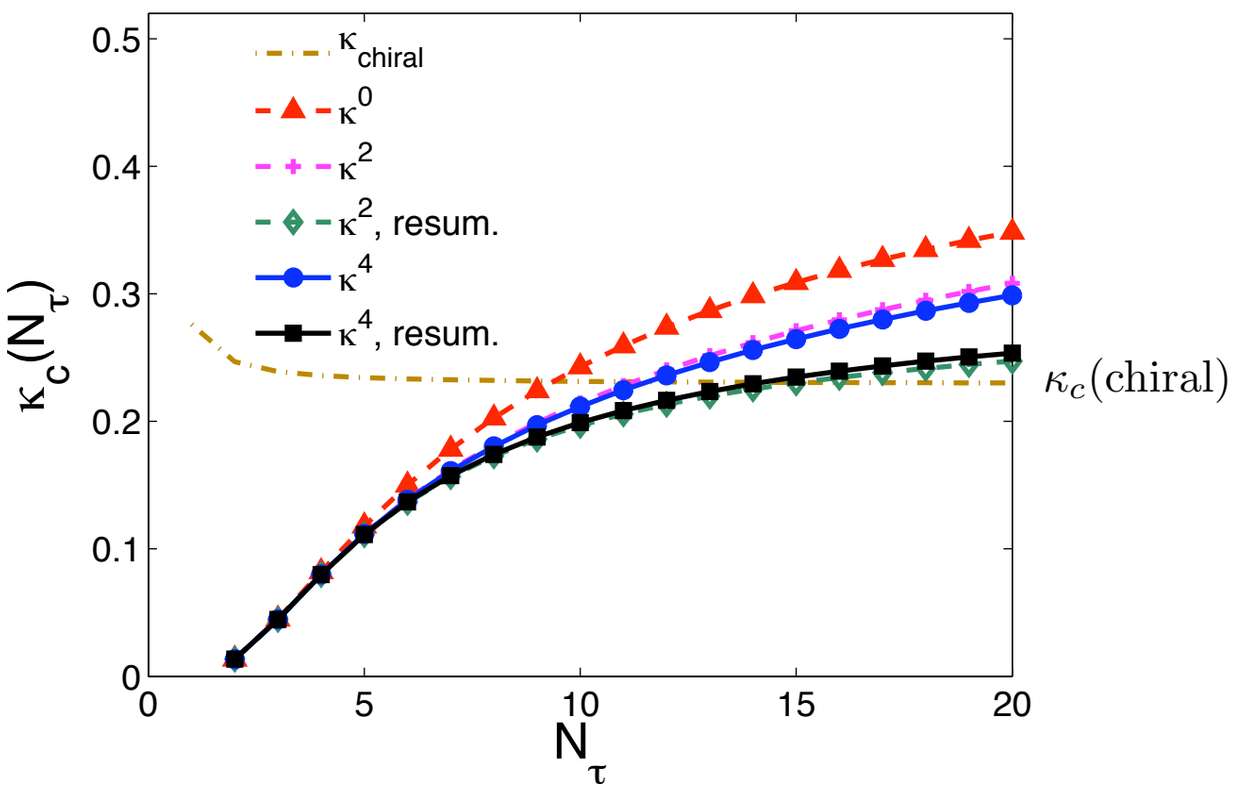
$$\lambda_c = 0.18672(7), h_c = 0.000731(40)$$

Mapping back to QCD:

N_f	M_c/T	$\kappa_c(N_\tau = 4)$	$\kappa_c(4)$, Ref. [23]	$\kappa_c(4)$, Ref. [22]
1	7.22(5)	0.0822(11)	0.0783(4)	~ 0.08
2	7.91(5)	0.0691(9)	0.0658(3)	–
3	8.32(5)	0.0625(9)	0.0595(3)	–

$e^{-M/T} \simeq h/N_f$ [linear approximation in $h \ll 1 \ldots$]

Convergence properties:



Finite density: sign problem!

- Metropolis algorithm: Mild sign problem; $\frac{\mu}{T} \lesssim 3$
- Worm algorithm: No sign problem cf. Gattringer et al.

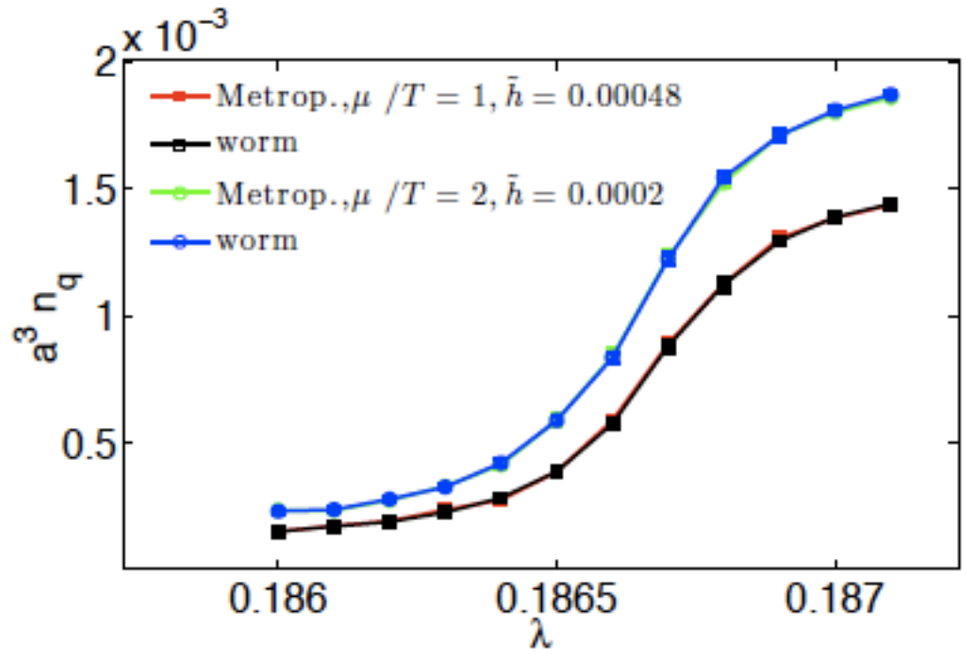
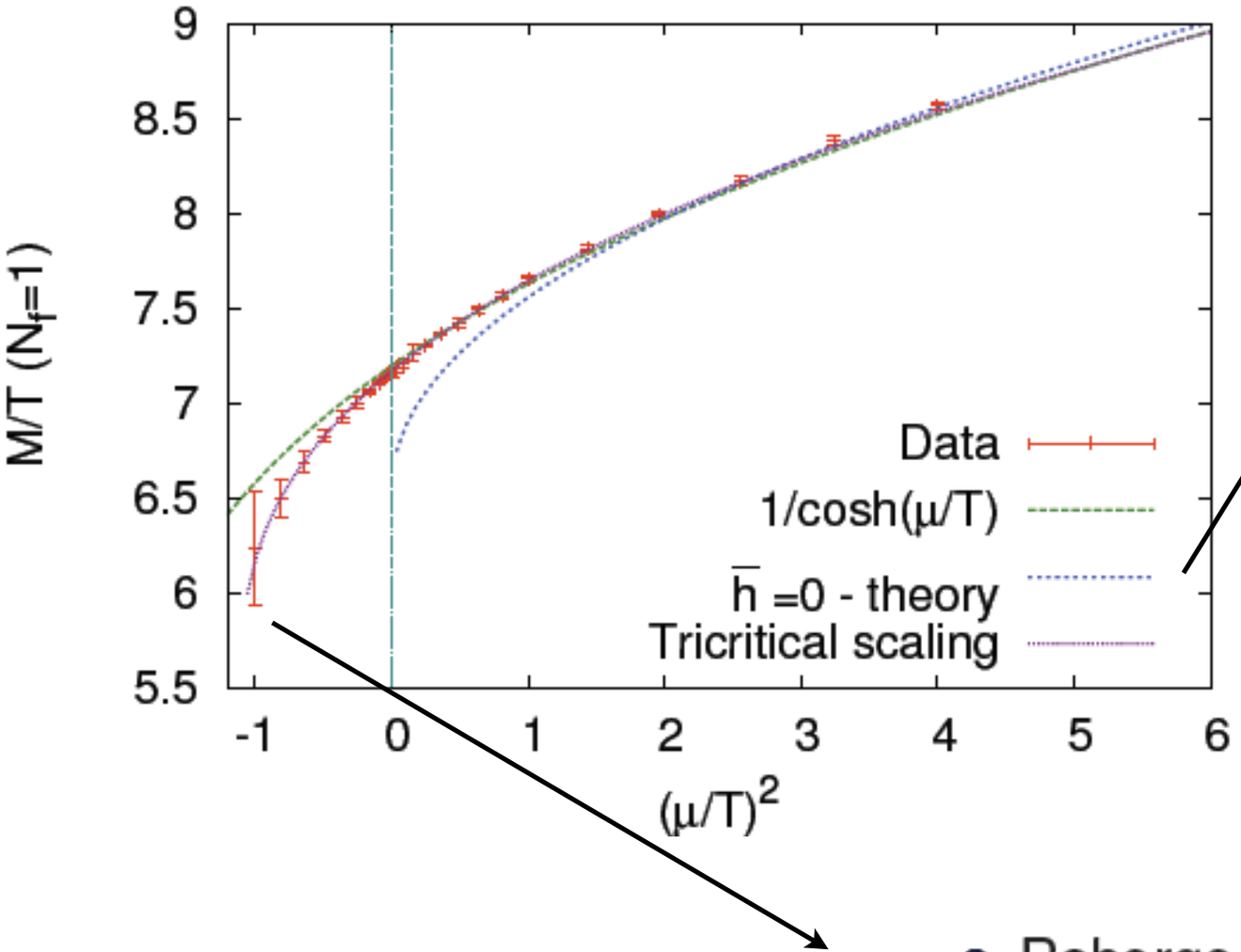


Figure: Quark density calculated with $Z_{\rm eff}$ from Metropolis or worm algorithm on 24^3 lattices for $\frac{\mu}{T} = 1$ and 2.

Critical quark mass as function of chemical potential

dense massive limit $\kappa \to 0, \mu \to \infty, \kappa e^{\mu/T} = \text{constant}$



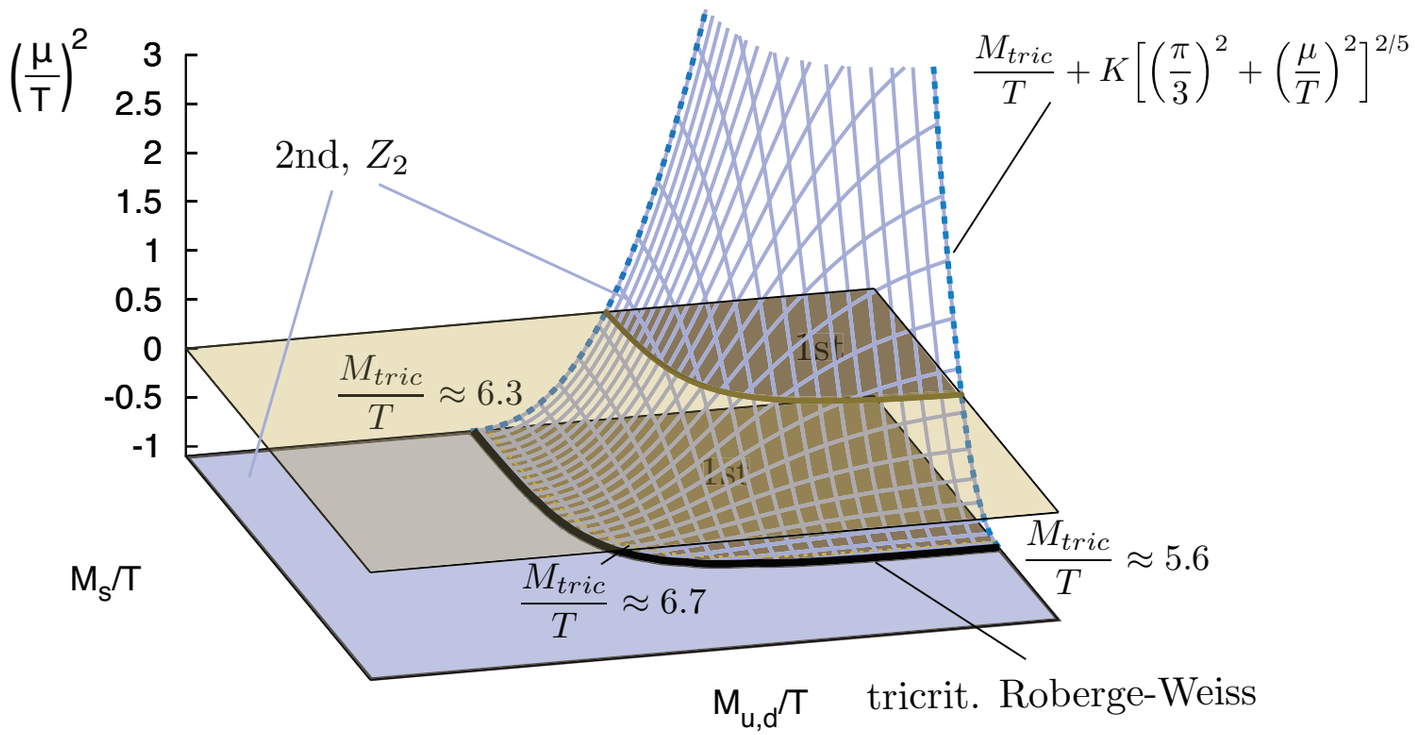
de Forcrand, O.P. 10
D'Elia, Sanfilippo 10

- Roberge-Weiss (tricritical) endpoint at $\mu_i/T = \pi/3$
  ($\leftrightarrow$ boundary between Z_3-sectors)

Tricritical scaling works perfectly, even well into $\mu^2 > 0$!

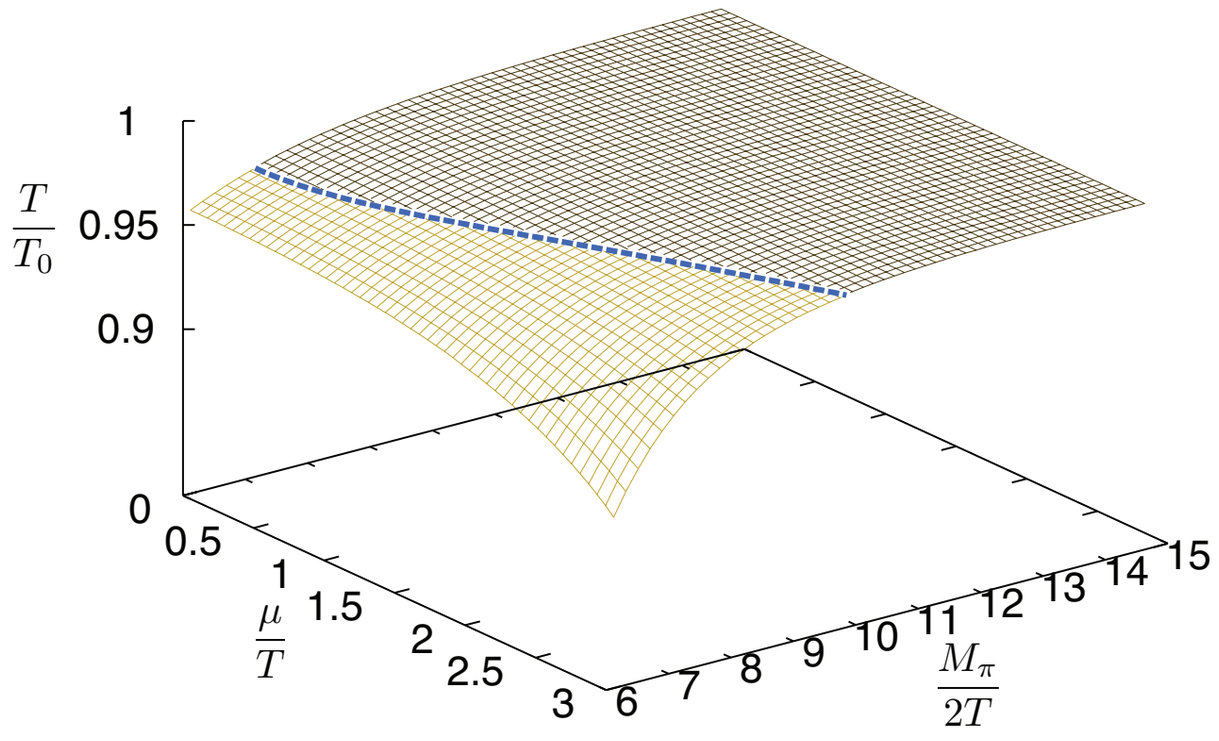
$$\frac{M_c}{T} = \frac{M_{\text{tric}}}{T} + K\left[\left(\frac{\pi}{3}\right)^2 + \left(\frac{\mu}{T}\right)^2\right]^{2/5}$$

The fully calculated deconfinement transition



deconfinement critical surface

phase diagram for Nf=2, Nt=6



Cold and dense QCD

Z(3) breaking part (fermion determinant) including second coupling

$$e^{-S^a_{\rm eff}[W]} = \prod_n \Delta_n[W]$$

$$\Delta_1 \;=\; \prod_{f,i} \det[1 + h_{1f} W_i]^2 [1 + \bar{h}_{1f} W_i^\dagger]^2$$

includes qq interaction!

$$\Delta_2 \;=\; \prod_{f,<ij>} \left[1 - h_{2f} N_\tau \mathrm{Tr}_c \frac{W_i}{1 + C_f W_i} \mathrm{Tr}_c \frac{W_j}{1 + C_f W_j}\right]^2$$

$$h_{1f} \;=\; C_f \left[1 + 6\kappa_f^2 N_\tau \frac{u - u^{N_\tau}}{1 - u} + \ldots\right] \qquad h_{2f} \;=\; C_f^2 \frac{\kappa_f^2}{N_c} \left[1 + 2\frac{u - u^{N_\tau}}{1 - u} + \ldots\right]$$

$$C_f \equiv (2\kappa_f e^{a\mu_f})^{N_\tau} \;=\; e^{(\mu_f - m_f)/T},\; \bar{C}_f(\mu_f) \;=\; C_f(-\mu_f)$$

Cold and dense QCD I: static, strong coupling limit

For T=0 (at finite density) anti-fermions decouple $N_f = 1, h_1 = C, h_2 = 0$

$$C_f \equiv (2\kappa_f e^{a\mu_f})^{N_\tau} = e^{(\mu_f - m_f)/T},\ \bar{C}_f(\mu_f) = C_f(-\mu_f)$$

$$Z(\beta = 0) = \left[\prod_f \int \mathrm{d}W \left(1 + C_f L + C_f^2 L^* + C_f^3\right)^2 \right]^{N_s^3}$$

$$\xrightarrow{T\to 0} \left[1 + 4C^{N_c} + C^{2N_c}\right]^{N_s^3}$$

Free gas of baryons!

Quarkyonic?

$$n = \frac{T}{V}\frac{\partial}{\partial \mu} \ln Z = \frac{1}{a^3} \frac{4N_c C^{N_c} + 2N_c C^{2N_c}}{1 + 4C^{N_c} + C^{2N_c}}$$

$$\lim_{\mu\to\infty} (a^3 n) = 2N_c$$

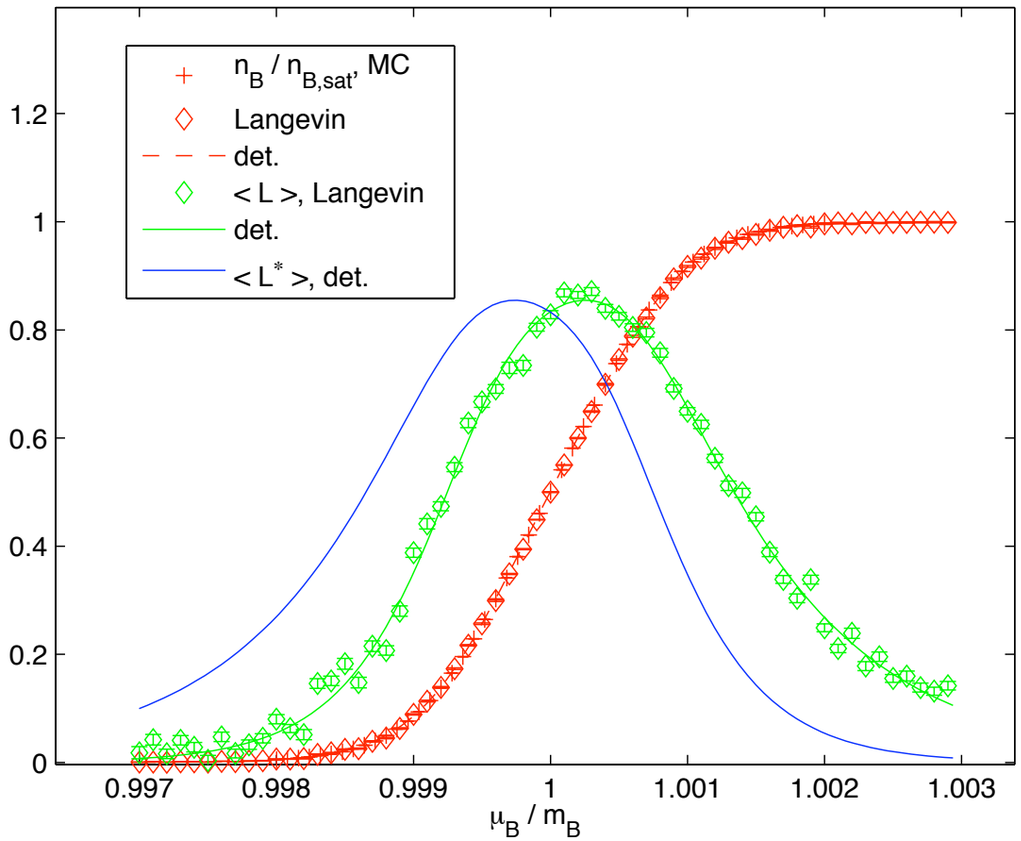
Sivler blaze + saturation, Pauli principle!

$$\lim_{T\to 0} a^3 n = \begin{cases} 0, & \mu < m \\ 2N_c, & \mu > m \end{cases}$$

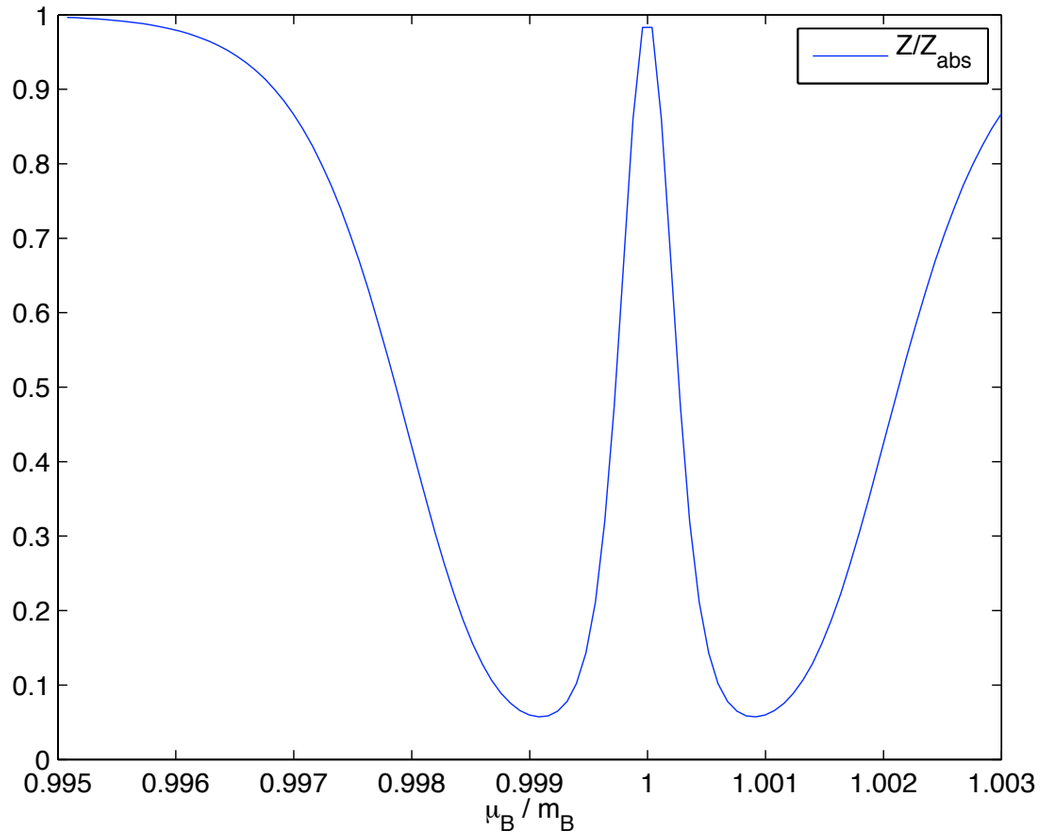
Cold and dense QCD II: interacting, dynamical!

$$m_\pi = 20\ \mathrm{GeV}, T = 10\ \mathrm{MeV}, a = 0.17\ \mathrm{fm}$$

$$\beta = 5.7, \kappa = 0.0000887, N_\tau = 116$$



Metropolis, average sign:



Analytic strong coupling sol. valid:

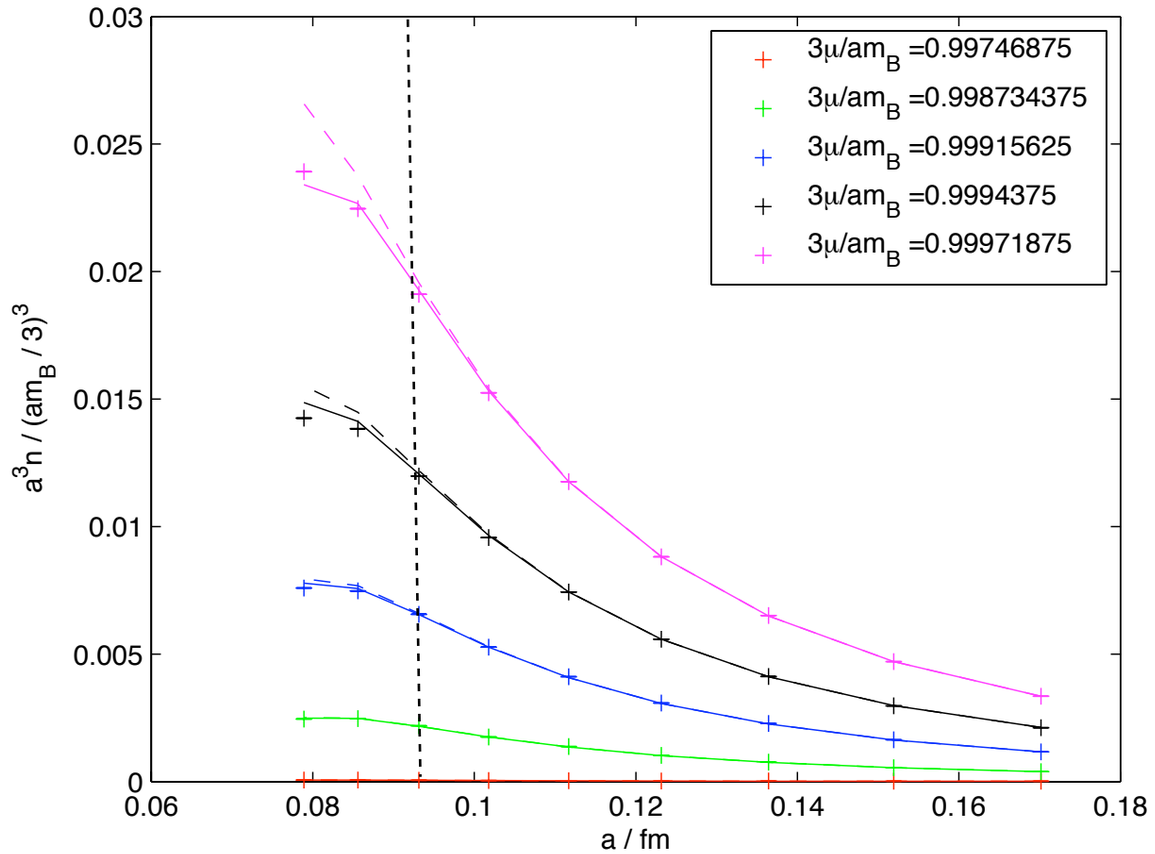
$$\lambda(\beta = 5.7, N_\tau = 115) \sim 10^{-27}$$

Complex Langevin: convergence criteria satisfied, cf. Aarts, James 12

Continuum extrapolation

Scaling with lattice spacing:

$$\frac{n_{\rm lat}(\mu)}{m_B^3} = \frac{n_{\rm cont}(\mu)}{m_B^3} + A(\mu)a + B(\mu)a^2 + \dots$$

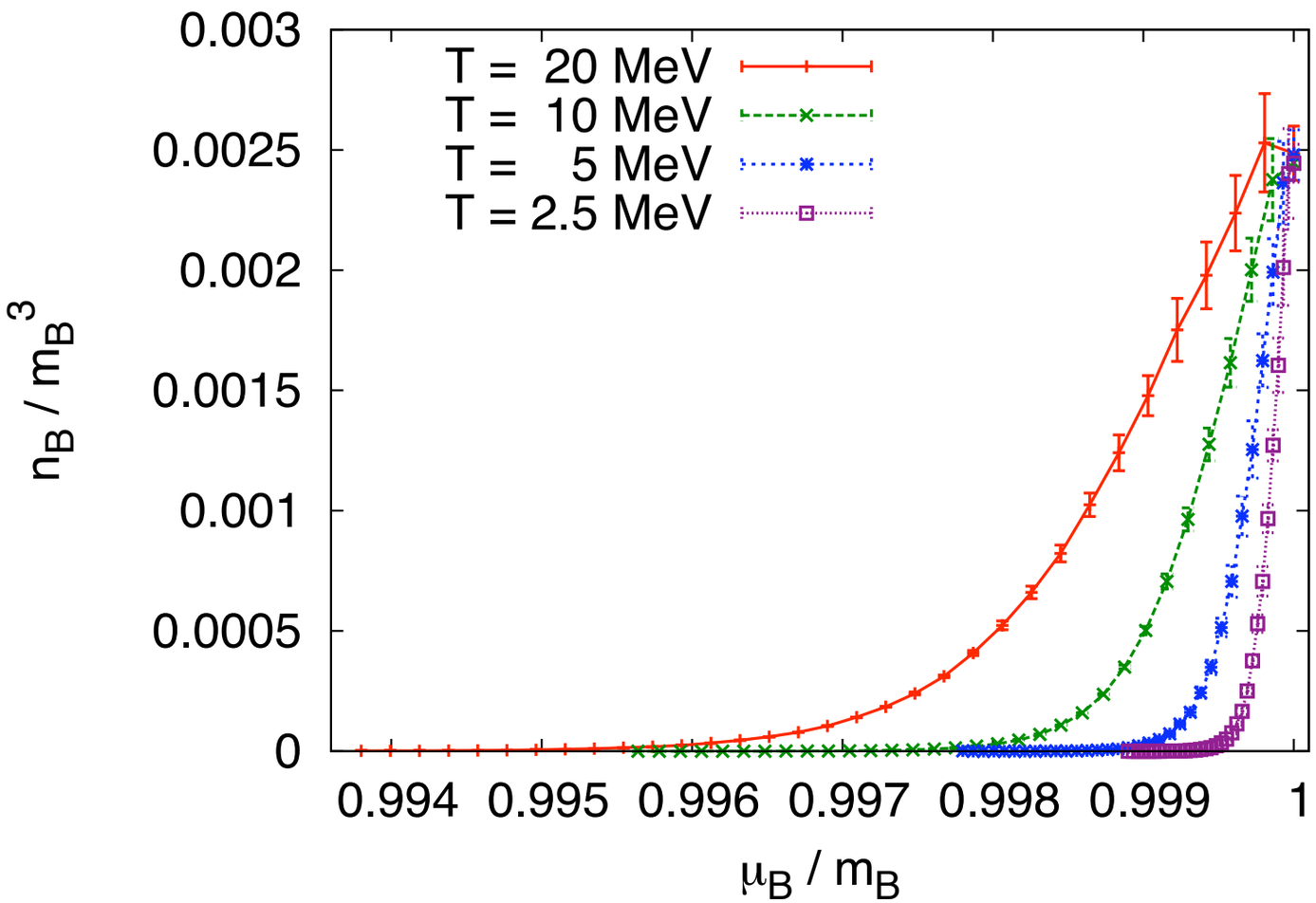


Solid/dashed lines: analytic strong coupling limit with/without $\mathcal{O}(\kappa^2)$:

Breakdown of hopping series!

The silver blaze property of QCD in the continuum

... with very heavy quarks



consistent with nuclear density!

Conclusions

- Two-step treatment of QCD phase transitions:

  I. Derivation of effective action by strong-coupling expansion
  II. Simulation of effective theory

- Z(N)-invariant effective theory for Yang-Mills, correct order of p.t., Tc with better than 10% accuracy in the continuum limit

- Finite T deconf. transition for heavy fermions and all chemical potentials

- Sivler blaze property + 1st. order phase transition to nuclear matter at T=0