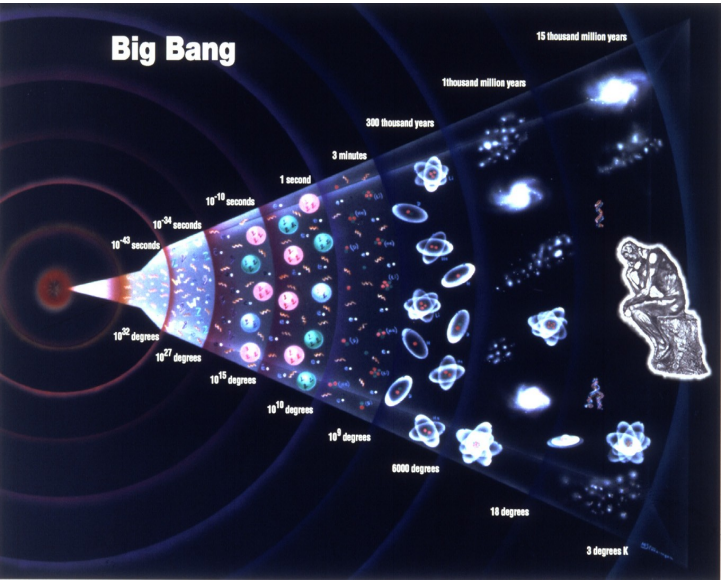
Turbulence and Bose-Einstein condensation Far from Equilibrium

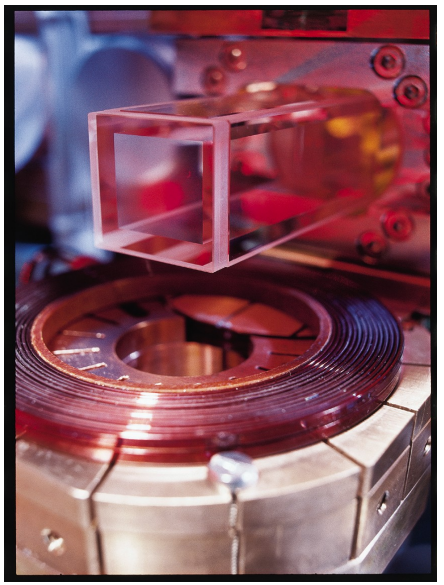
Dénes Sexty

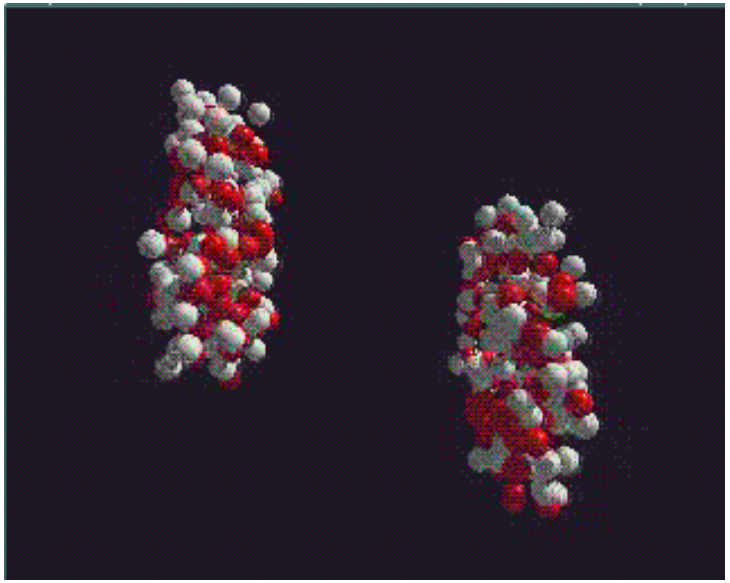
Uni Heidelberg

Collaborators: Jürgen Berges, Sören Schlichting

July 12, 2012 , Swansea
SEWM 2012







Non-equilibrium initial state



Turbulent flow



Thermal equilibrium

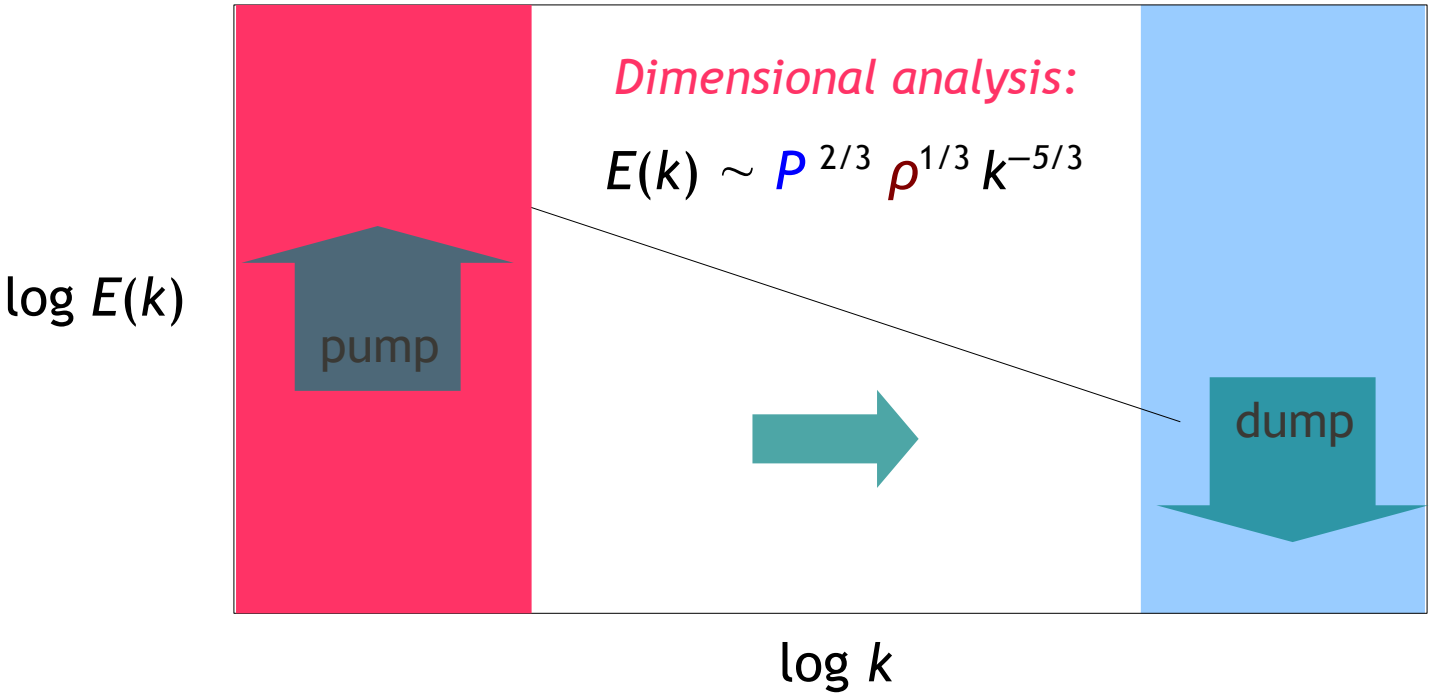


Kolmogorov turbulence

"local" interactions in momentum space

Constant flux in momentum space

Scale invariant transport



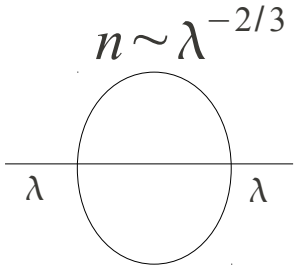
Turbulence in an incompressible fluid

3D:	Radial energy density	E	[kg s^{-2}]
	Radial energy flux	P	[kg m^{-1} s^{-3}]
	Density	ρ	[kg m^{-3}]

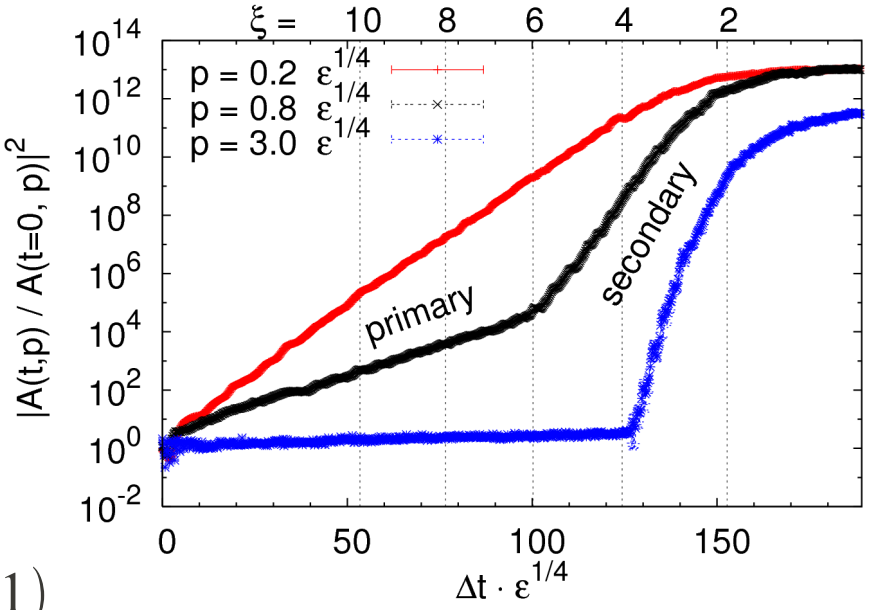
“Natural” nonequilibrium initial conditions

Instability for some momentum band

Amplitude grows

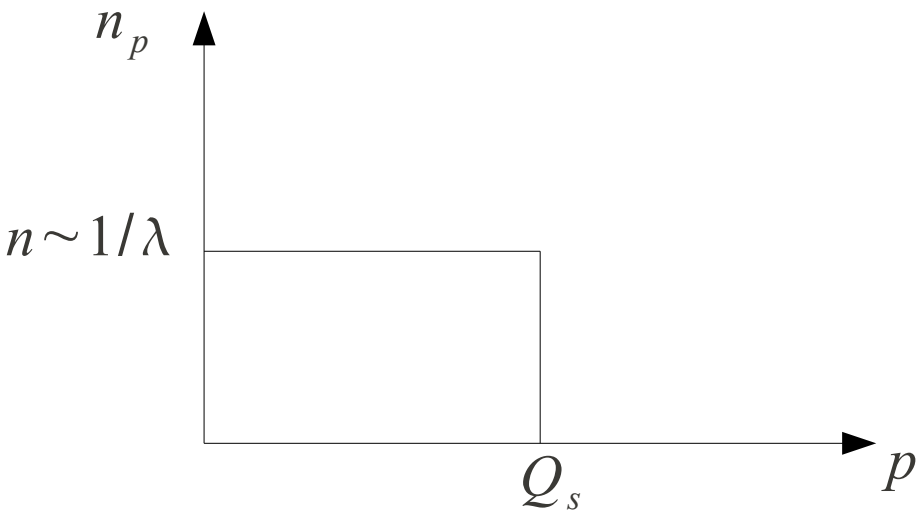


Secondary modes



At $n \sim 1/\lambda$ all diagrams become $\sim O(1)$

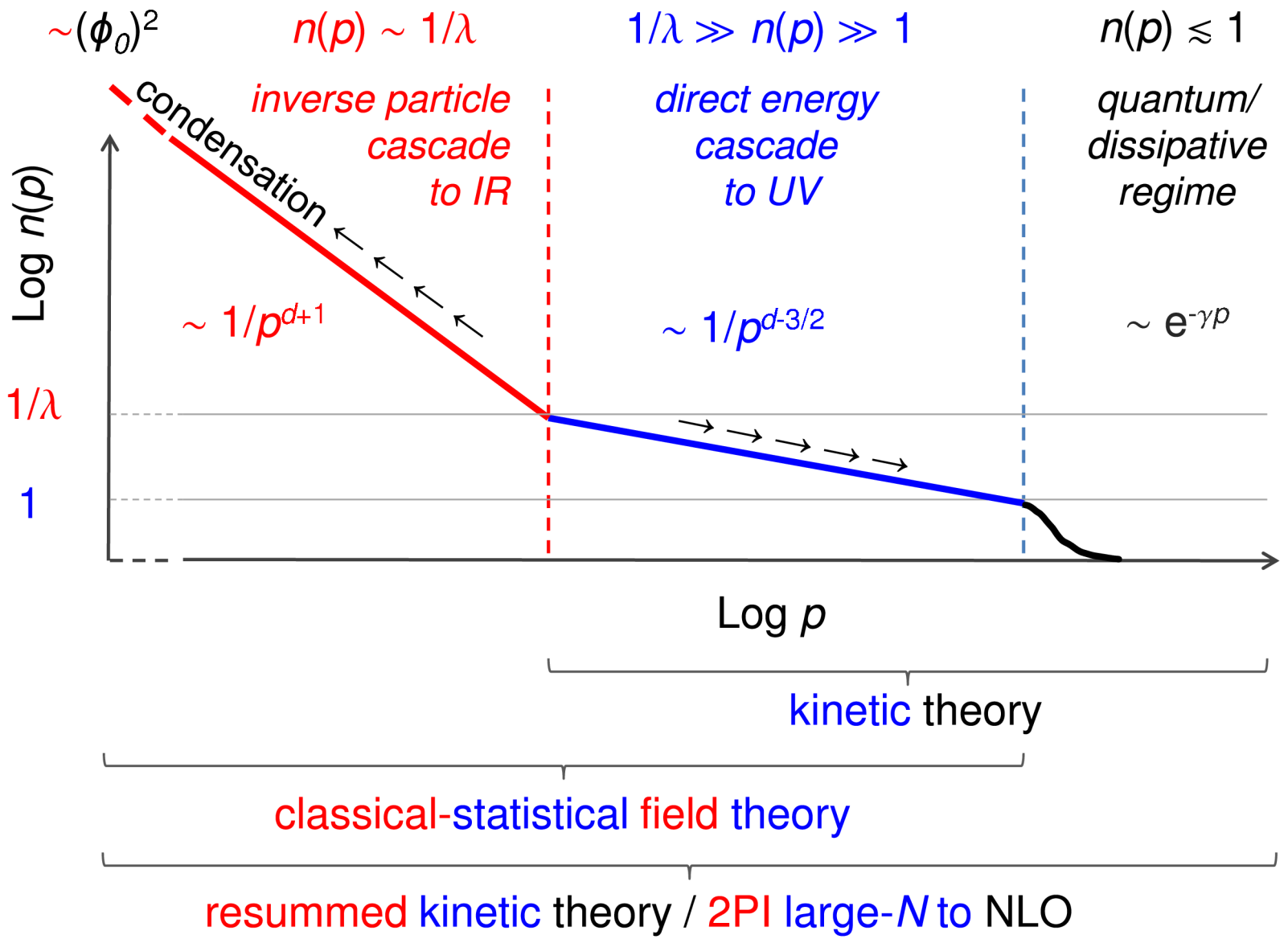
Non-linear physics; instability saturates



Overoccupation

Tachyonic reheating
Parametric reheating
Weibel instability
CGC

Turbulence and condensation in scalar field theories



Bose-Einstein Condensation and Thermalization of the Quark Gluon Plasma

[Blaziot et al, 2011]

Initial CGC: $\epsilon_0 \sim \frac{Q_s^4}{\alpha_s} \qquad n_0 \sim \frac{Q_s^3}{\alpha_s} \quad \rightarrow \quad n_0 e_0^{-3/4} \sim \alpha_s^{-1/4}$

Thermal eq: $\epsilon_{eq} \sim T^4 \qquad n_{eq} \sim T^3 \quad \rightarrow \quad n_0 e_0^{-3/4} \sim 1$

Elastic processes dominate

Particles pile up in the IR

Overpopulation leads to emergence of condensate

What is a condensate?

In equilibrium: $N = V \int \frac{d^3k}{2\pi^3} \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$ Maximum at $\mu = \epsilon_0$

Condensation: $N > N_{max}$

Macroscopic occupation of the zero mode Condensate fraction $\frac{N_0}{N}$

Particle distribution: $n_k = \delta^3(k) n_0 + n'_k$

In terms of 2point function $F(x,y) = \{\phi(x), \phi(y)\}$

$$n_k = F(k)\omega_k \Rightarrow \qquad F(k=0) \sim V$$

condensate $= \left| \frac{\int d^3x \phi(x)}{V} \right|^2 = \frac{F(k=0)}{V}$ Independent of the volume

Condensation in bose gas

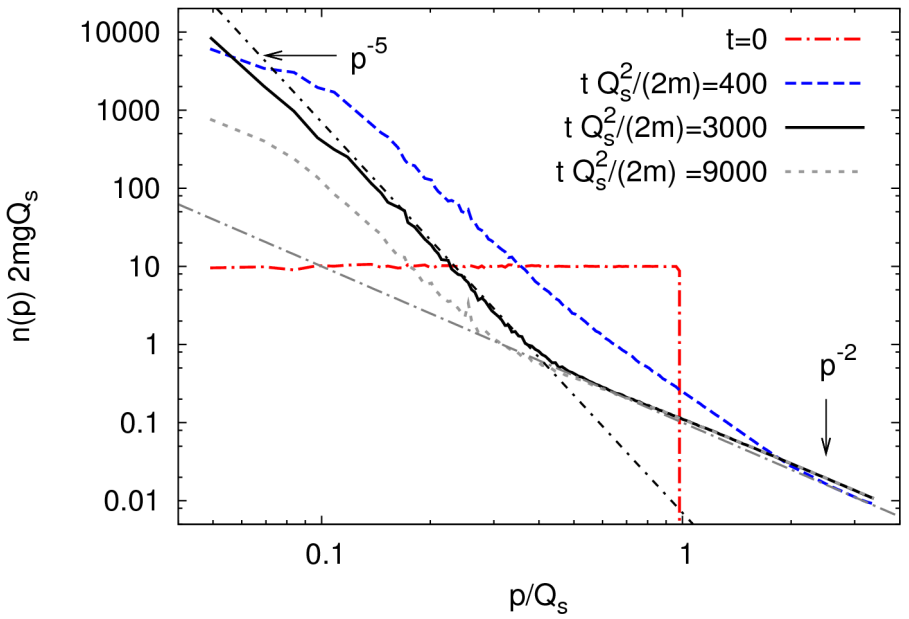
[Berges, Sexty (2012)]

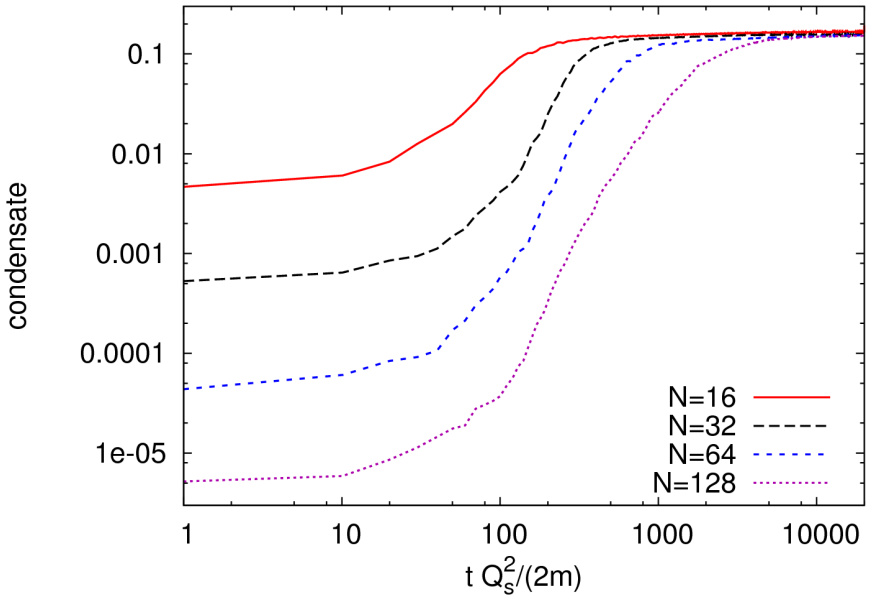
Non relativistic scalars described by complex field $\Psi(x,t)$

Gross-Pitaevski equation: $$i\partial_t \Psi(x,t) = \left(-\frac{\partial_i^2}{2m} + g|\Psi(x,t)|^2\right)\Psi(x,t)$$

conserved particle number $$n_{tot} = \int d^3x |\Psi(x,t)|^2$$

occupation in zero mode: condensate = $$\left|\frac{\int d^3x \Psi(x,t)}{V}\right|^2$$





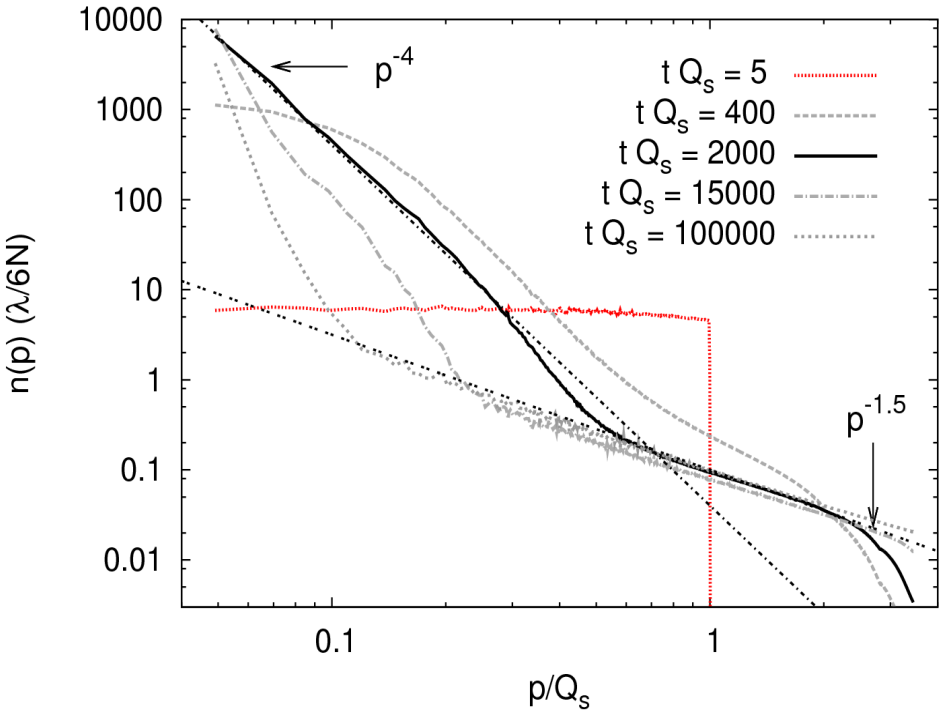
Non-equilibrium Bose condensation

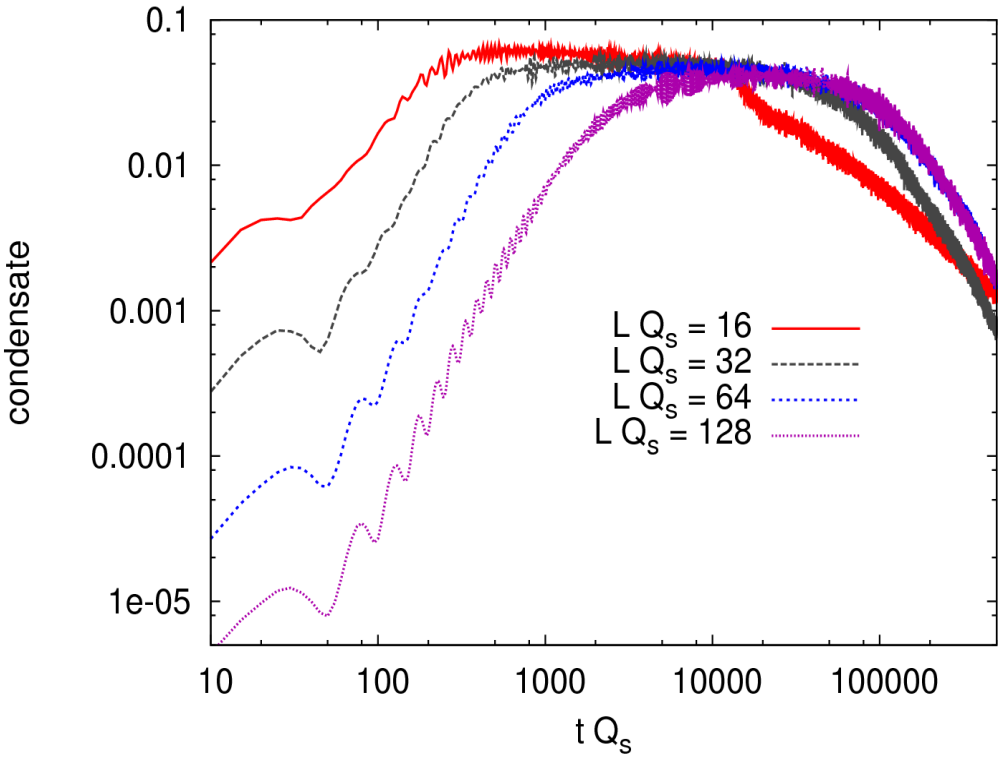
[Berges, Sexty (2012)]

O(4) massless relativistic scalars

Initial conditions: overpopulation

condensate $= \left\langle \left| \frac{\int d^3x \, \phi_a(x)}{V} \right|^2 \right\rangle_{ens}$





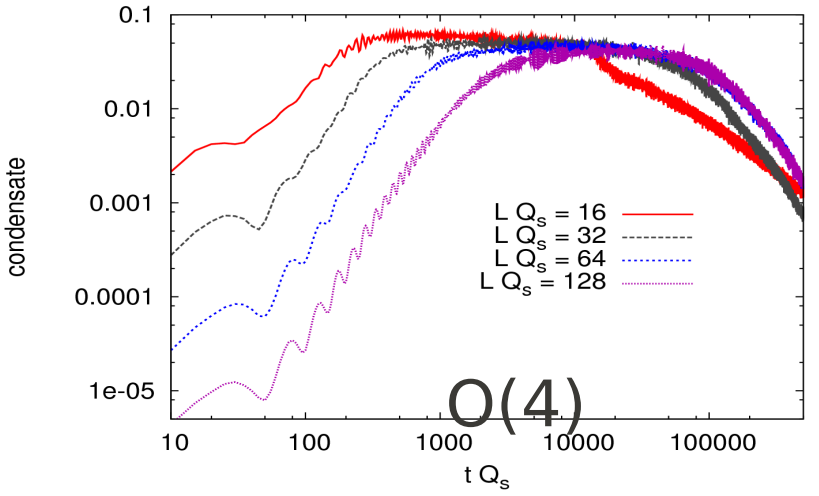
Decay of the condensate

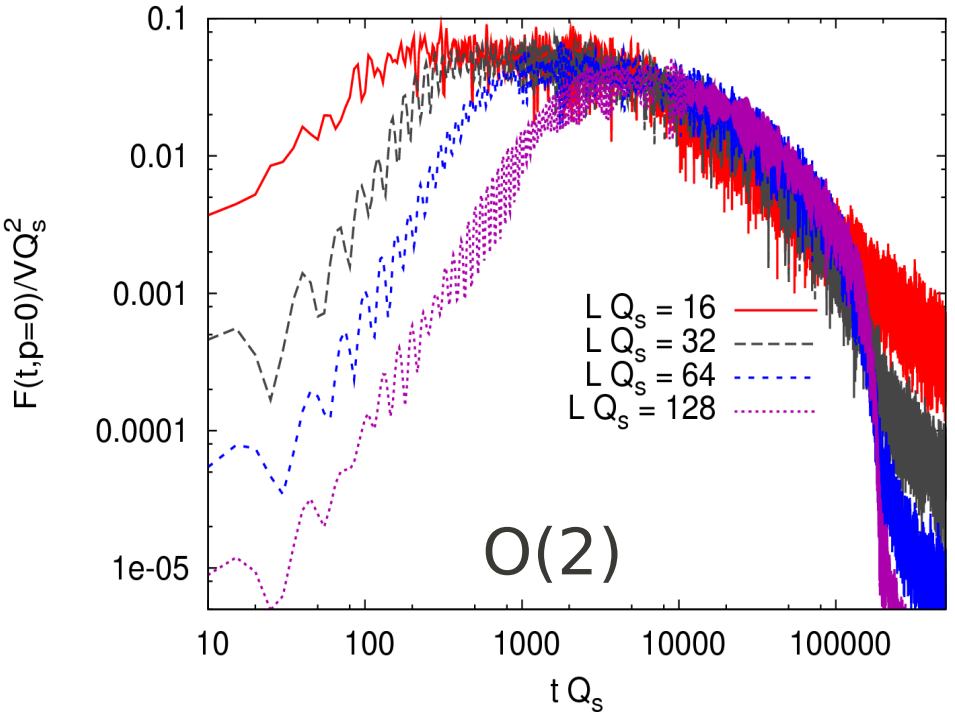
Particle number changing processes suppressed → condensation

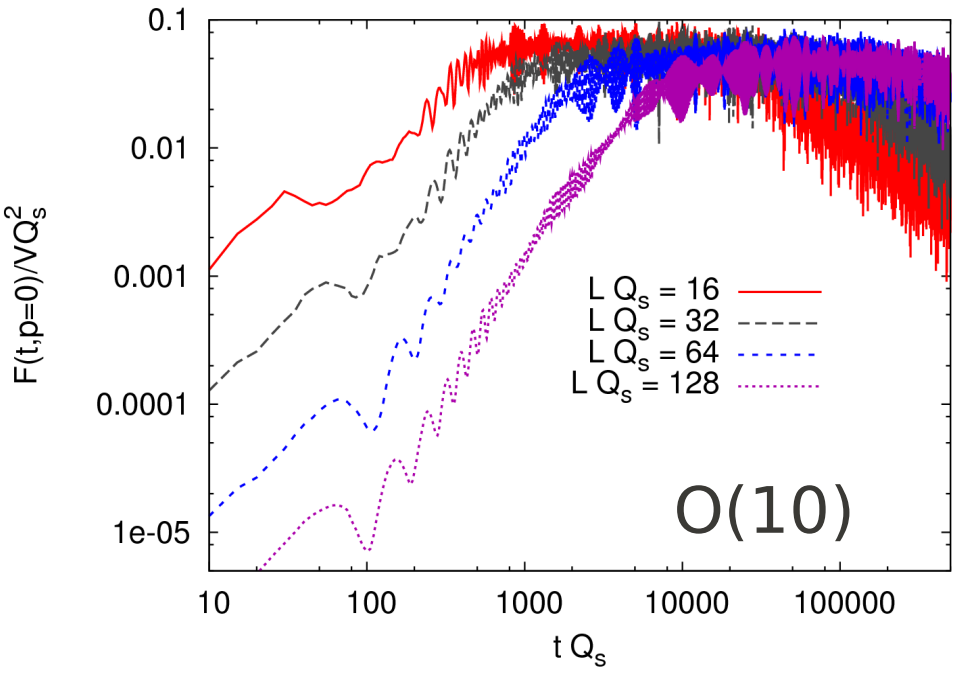
↓

decay of condensate

Suppression compared to 2->2 $\sim \frac{1}{N}$







Turbulent cascade

Conserved charge

$$\partial_k n_k = 0$$

→ Stationary power law solution with k-independent flow

2->2 dominates: particle number effectively conserved

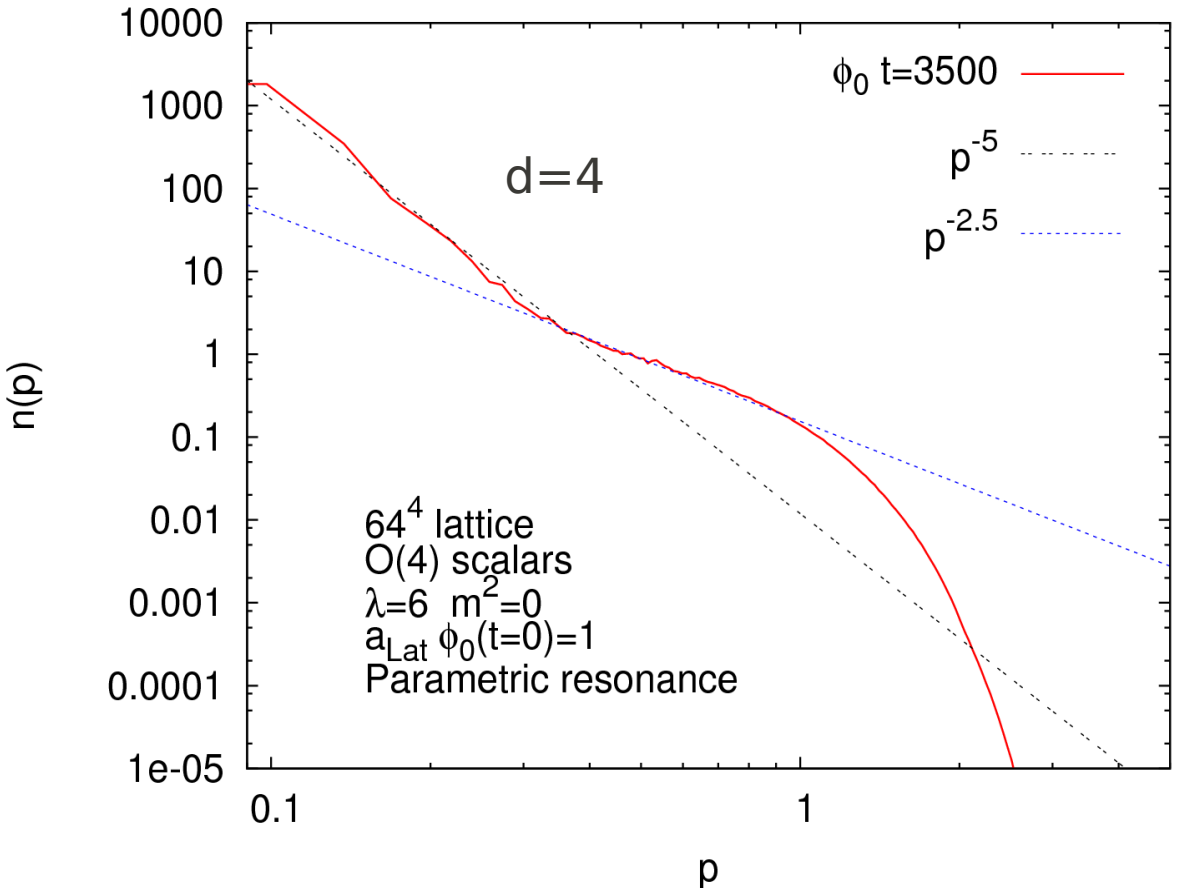
Dual cascade: particles to IR
energy to UV

Particle flow Energy flow

$$\kappa_{IR} = d+1 \text{ or } \kappa_{IR} = d+2$$

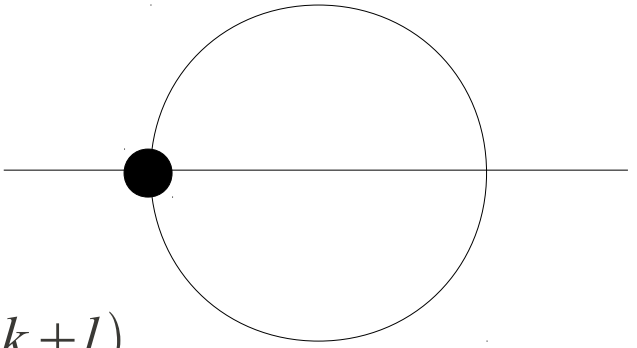
$$\kappa_{UV} = d-2 \text{ or } \kappa_{UV} = d-3/2$$

$$n_k \sim k^{-\kappa}$$



IR resummation – Strong turbulence

1/N resummation: effective vertex



$$\Sigma(p)=\int_{kql}\lambda_{eff}(p+q)G(q)G(k)G(l)\delta^{(4)}(p+q+k+l)$$

$$\lambda_{eff}(p)=\frac{\lambda}{(1+\Pi^{R}(p))(1+\Pi^{A}(p))}$$

With one loop bubble:

$$\Pi(p)=\int_{q}G(p)G(p-q)$$

In the IR: $\Pi(p)\gg 1$

The vertex scales:

$$\lambda_{eff}(s\,p)=s^{2r}\lambda_{eff}(p) \text{ with } r=3+\kappa-d$$

In the UV: $\lambda_{eff}=\lambda$

$$\lambda(sp)=\lambda(p)$$

Strong turbulence in the IR: $\kappa=4$ or 5 (in d=3)

From 2PI to kinetic equations

Using Wigner coordinates

$$F_p(X)=\int d^4 s \exp(-ip_\mu s^\mu) F(X+s/2, X-s/2)$$

Gradient expansion, spatially homogeneous ensemble:

$$\partial_t \rho_p(X)=0$$

$$2\,p_0\,\partial_t F_p(X)=\Sigma^\rho_p(X)F_p(X)-\Sigma^F_p(X)\rho_p(X)$$

Define:

$$F_p(X)=(n_p(X)+1/2)\rho_p(X)$$

$$n_{eff}(t,\boldsymbol{p})=\int_0^\infty \frac{dp_0}{2\pi} 2\,p_0\,\rho_p(X)n_p(X)$$

On-shell limit, only 2->2 contributes

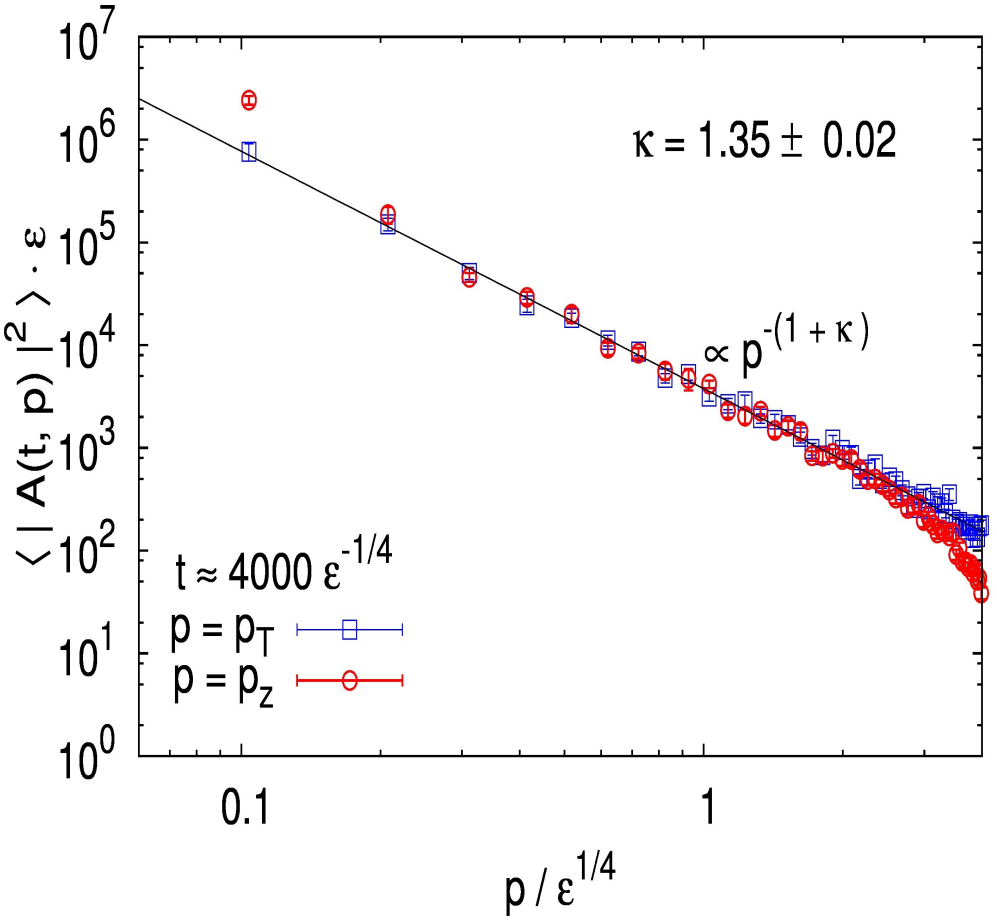
$$\partial_t n_{eff}(t,\boldsymbol{p})=\int d\Omega_{2\to 2}\Big[(1+n_p)(1+n_l)n_q n_r-n_p n_l(1+n_q)(1+n_r)\Big]\lambda_{eff}(p+l)$$

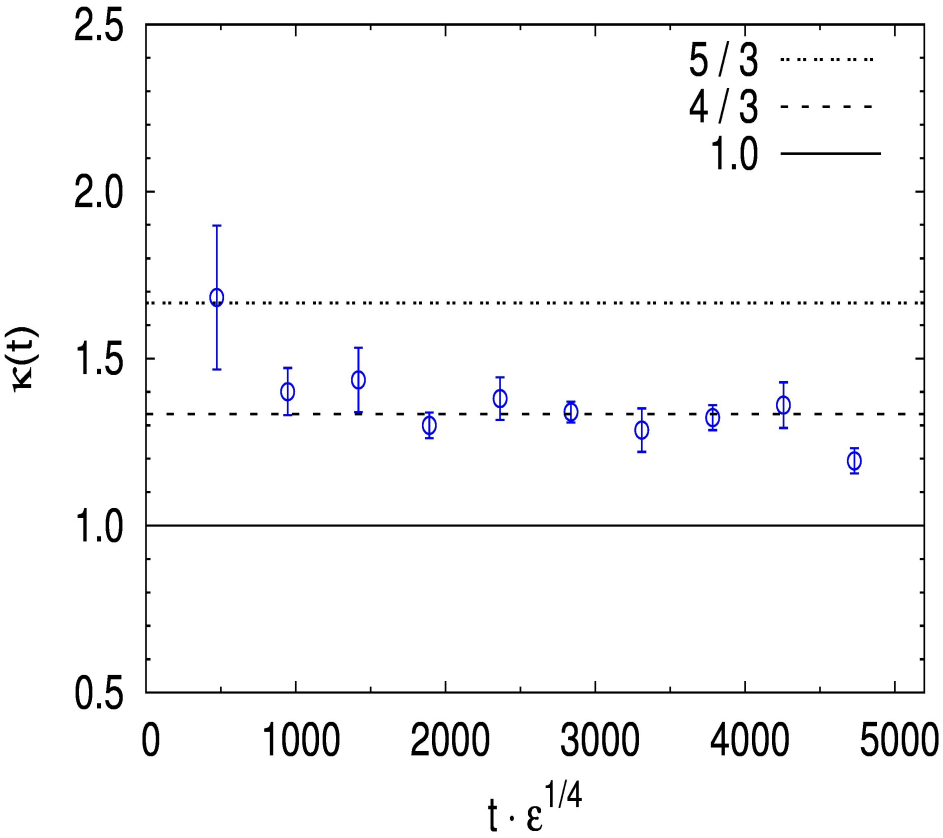
Effective kinetic description also valid at $\lambda n \gtrsim 1$

[Berges, Sexty (2011)]

UV cascade in gauge theories

SU(2) gauge theory after Weibel instability and isotropisation





[Berges, Scheffler, Sexty (2009)]

See also [Kurkela, Moore arXiv:1207.1663]
[Schlichting arXiv:1207.1450]

Weak wave turbulence

[Micha, Tkachev (2004)]

Power law distributions $n(k) \sim k^{-\kappa}$

Self similar evolution: $n(k,t) \sim t^{-\gamma p} n_0(k t^{-p})$

Fluctuation decay $\langle \varphi^2 \rangle_{conn} \sim t^{-\nu}$

$\dot{n}(k) = I[n(k)]$

Interaction through m-vertex

k-independent stationary flow

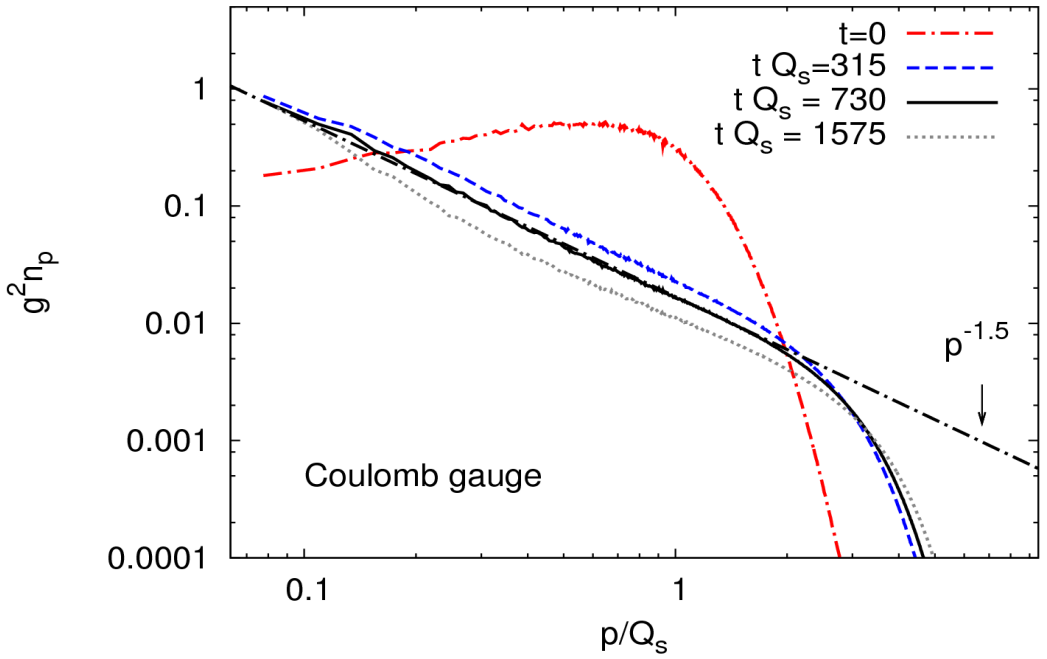
Particle cascade
Energy cascade

$$\mu = d(m-2) - 1 - m \qquad \kappa = \frac{d+\mu}{m-1} \qquad \gamma = d \qquad p = \frac{1}{2m+1} \qquad \nu = \frac{2}{2m-1}$$

Universality far from equilibrium

Gauge theory UV turbulence

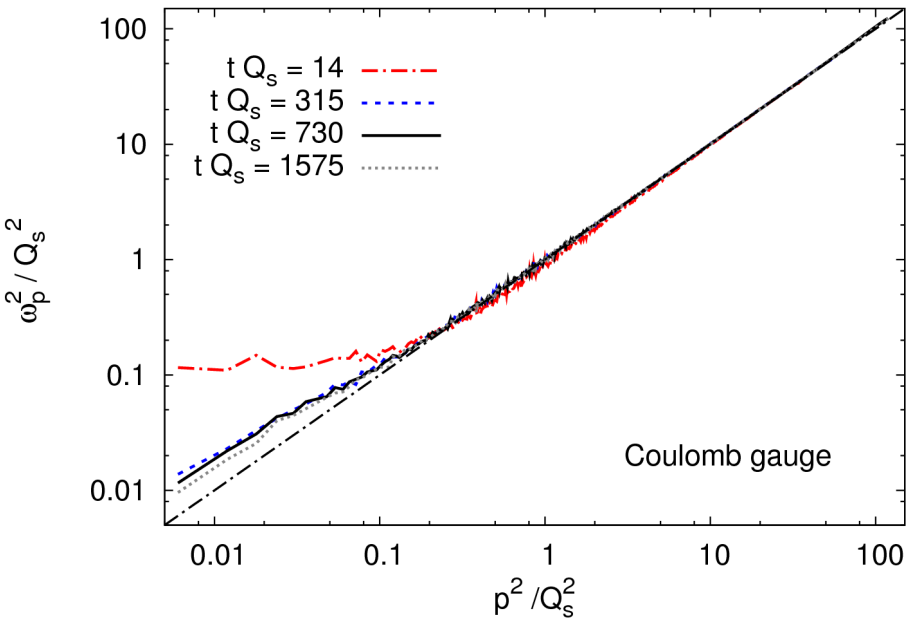
Pure SU(2) gauge theory overpopulated initial condition



$$n_p \sim p^{-1.5}$$

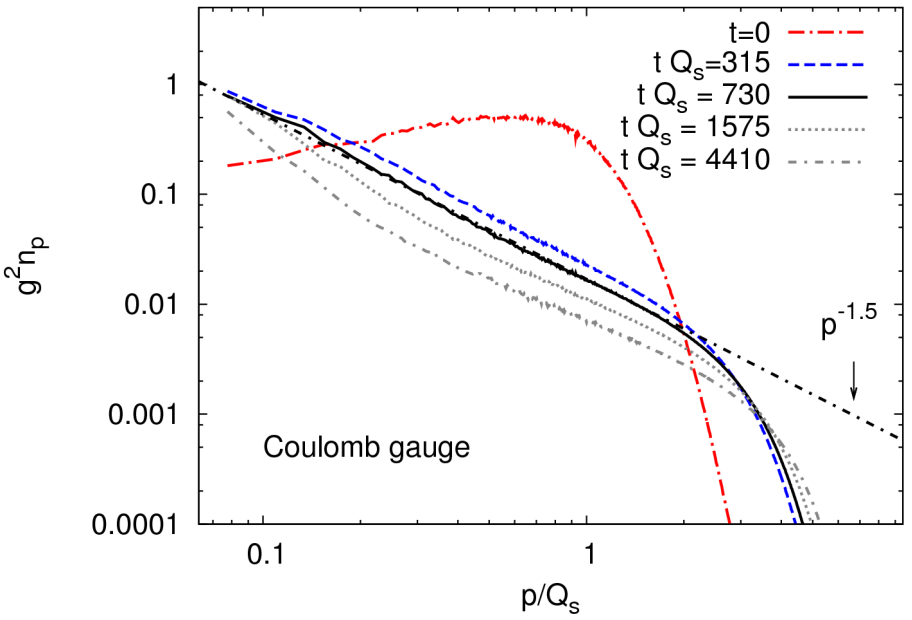
same as scalar UV exponent

Dispersion



[Berges, Schlichting, Sexty, arXiv:1203.4646]

Time dependence of gauge theory exponent

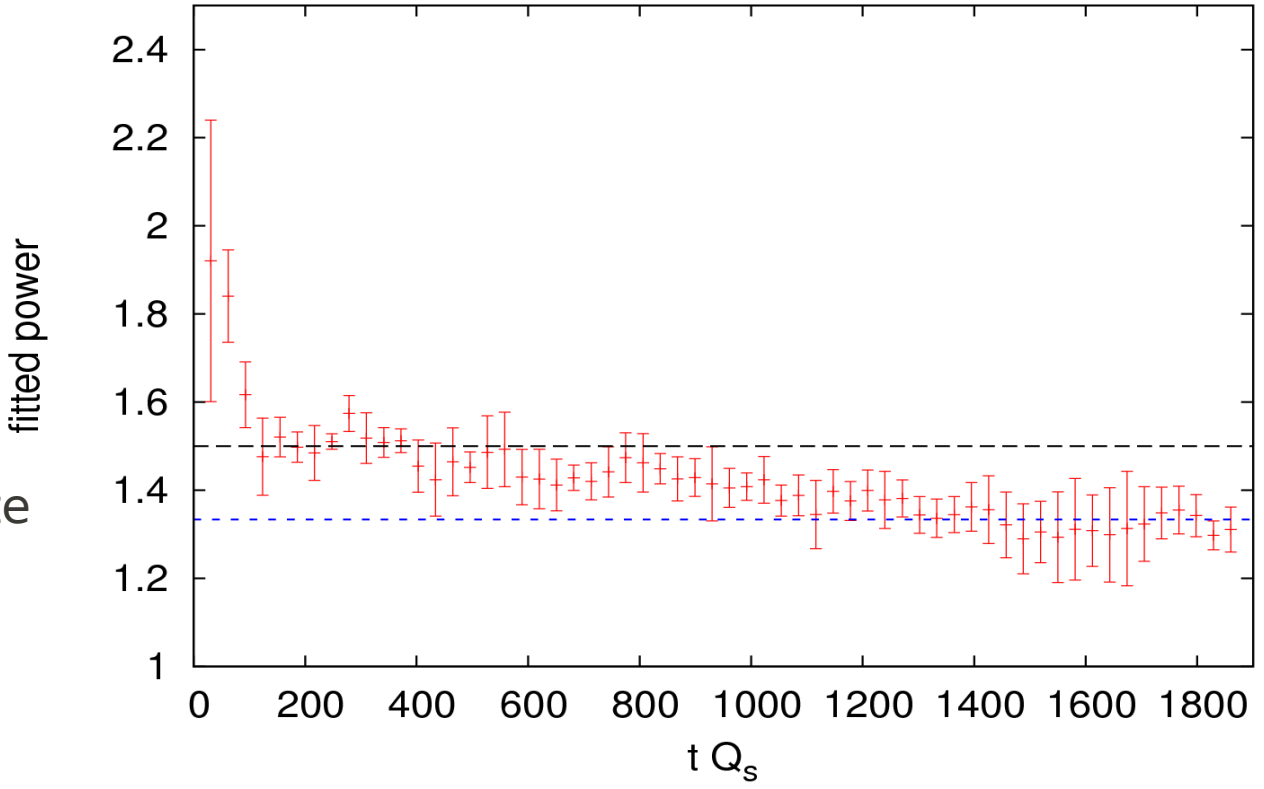


Fit $n(p) \sim p^{-\kappa}$ [0.4: 1.0]

Weak wave turbulence:

$\kappa = 3/2$ With a condensate

$\kappa = 4/3$ Without a condensate



Wave Turbulence

In terms of corrleation functions

$$\phi = \phi_a \quad \text{or} \quad A^a_\mu$$

$$F(x,y) = \{\phi(x), \phi(y)\}$$

$$\rho(x,y) = [\phi(x), \phi(y)]$$

Stationarity condition:

(Collision integral vanishes)

$$\Pi_\rho(p) F(p) - \Pi_F(p) \rho(p) = 0$$

With self energy: $\Pi(p)$

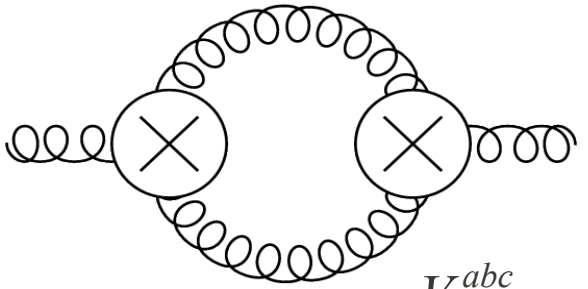
Scaling ansatz

$$F(s^z \omega, s\, p) = |s|^{-2-\kappa} F(\omega, p)$$

$$\rho(s^z \omega, s\, p) = |s|^{2-\eta} \rho(\omega, p)$$

Classicality condition $F(p) \gg \rho(p)$

Lowest order contribution to self energy:



$$V^{abc}_{\mu\nu\gamma} \sim g$$

$$V^{abc}_{\mu\nu\gamma} = V^{abc}_{0,\mu\nu\gamma} + V^{abc}_{A,\mu\nu\gamma}$$

$$V^{abc}_{0,\mu\nu\gamma}(p,q,k) = g\, f^{abc}\left(g_{\mu\nu}(p-q)_\gamma + g_{\nu\gamma}(q-k)_\mu + g_{\gamma\mu}(k-p)_\nu\right)$$

$$V^{abc}_{A,\mu\nu\gamma}(x,y,z) = \left(C_{ac,bd}\, g_{\mu\nu} A^d_\gamma(x) + C_{ab,dc}\, g_{\nu\gamma} A^d_\mu(x) + C_{ab,cd}\, g_{\mu\nu} A^d_\nu(x)\right)$$

with $A^d_\delta(x) \sim 1/g$ background field

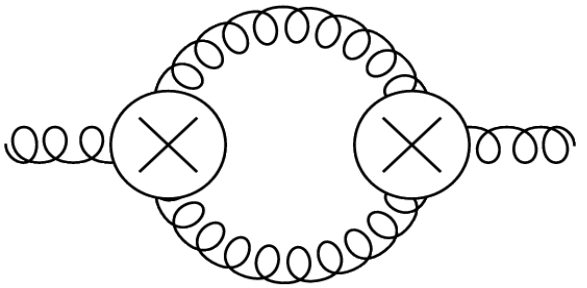
$$C_{ab,cd} = (f^{abe} f^{cde} + f^{ade} f^{cbe})\, g^2 \delta^{d+1}(x-y)\delta^{d+1}(x-z)$$

V_0 Kinematically forbidden on shell

Stationarity condition:

$$\Pi_F(p)\rho(p) - \Pi_\rho(p) F(p) = 0$$

Classical part of stationarity condinition:



$$\Pi_F \rho - \Pi_\rho F = \int_{pqk} (2\pi)^4 \delta^{(4)}(p+q+k) V^2$$
$$\left(\rho(p)F(q)F(k) + F(p)\rho(q)F(k) + F(p)F(q)\rho(k)\right)$$

Scaling ansatz:

$$F(sp) = |s|^{-(2+k)} F(p) \qquad \rho(sp) = |s|^{-2} \mathrm{sgn}(s)\rho(p) \qquad V(sp, sq, sk) = s^v V(p, q, k)$$

Transformation (swapping and rescaling)

$$q \to \frac{p_0}{k_0} q, \qquad k \to \frac{p_0}{k_0} p, \qquad p \to \frac{p_0}{k_0} k$$

$$\Pi_F \rho - \Pi_\rho F = \int_{pqk} (2\pi)^4 \delta^{(4)}(p+q+k) V^2$$
$$\rho(p)F(q)F(k)\left(1 + \left|\frac{p_0}{k_0}\right|^\Delta \mathrm{sgn}\left(\frac{p_0}{k_0}\right) + \left|\frac{p_0}{q_0}\right|^\Delta \mathrm{sgn}\left(\frac{p_0}{q_0}\right)\right)$$

Solution:

$$\Delta = -1 \qquad \kappa = \frac{3}{2}$$

Conclusions

Scalar case well understood

Dual cascade

Condensation

Weak and strong wave exponents
from kinetic theory (with resummation)

Gauge theory

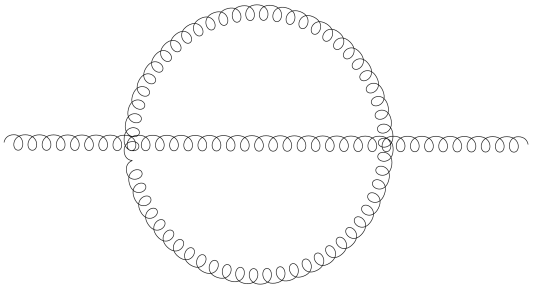
Numerical inditcation of scaling behaviour with $\kappa = 3/2$

(before 4/3 sets in)

May be explained with background field

g^2 contribution similar to scalars

Scaling analysis with sunset diagram



$$\Sigma(p)=\int_{qkl} G(q)G(k)G(l)\delta^{(4)}(p+q+k+l)$$

Classical part of the stationarity condition:

$$0=\int_{\boldsymbol{p}\,qkl} V(p,q,k,l)^2\delta^{(4)}(p+q+k+l)\big[F(p)F(q)F(k)\rho(l)+F(p)F(q)\rho(k)F(l)+F(p)\rho(q)F(k)F(l)+\rho(p)F(q)F(k)F(l)\big]$$

Zakharov transformation:
swapping momenta

$$l'=\xi p;\quad p'=\xi l;\quad k'=\xi k;\quad l'=\xi l$$

$$F(p)F(q)F(k)\rho(l)\Rightarrow\rho(p)F(q)F(k)F(l)$$

$$0=\int_{\boldsymbol{p}\,q\,k\,l} V(p,q,k,l)^2\delta^{(4)}(p+q+k+l)\rho(p)F(q)F(k)F(l)$$
$$\left[1+\left|\frac{p_0}{q_0}\right|^{\Delta}\operatorname{sgn}\left(\frac{p_0}{q_0}\right)+\left|\frac{p_0}{k_0}\right|^{\Delta}\operatorname{sgn}\left(\frac{p_0}{k_0}\right)+\left|\frac{p_0}{l_0}\right|^{\Delta}\operatorname{sgn}\left(\frac{p_0}{l_0}\right)\right]$$

Solutions:

$\Delta=-1$

$\Delta=0$ On shell limit 2->2 dominates

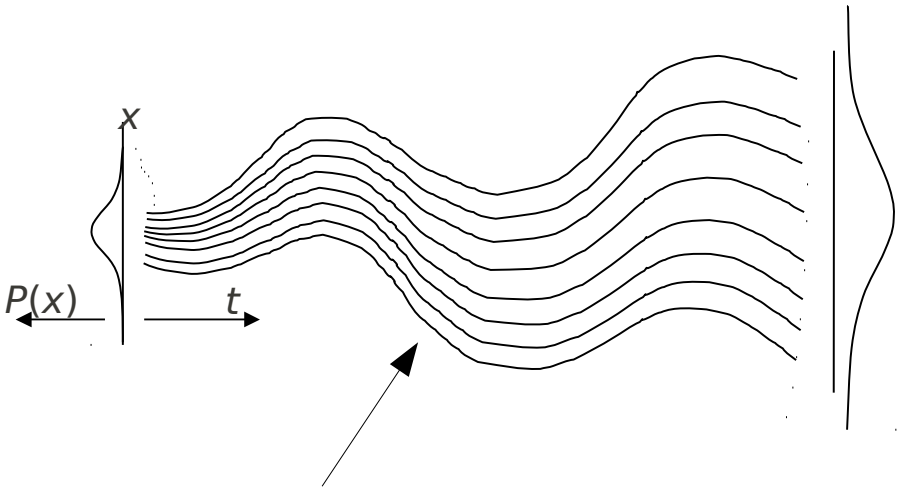
$$\kappa=\frac{5}{3}\ \text{and}\ \kappa=\frac{4}{3}$$

Classical statistical field theory

Classical fluctuations dominate over quantum for $n>1$

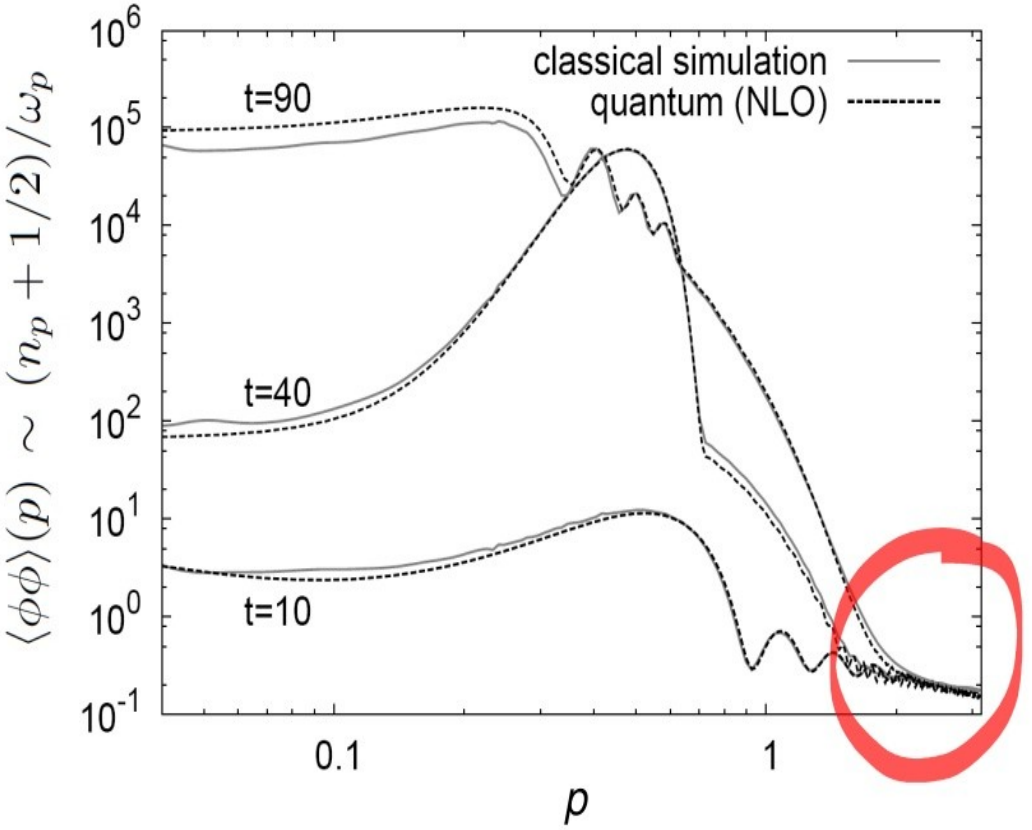
$$F=\langle\{\phi,\phi\}\rangle\gg\langle[\phi,\phi]\rangle=\rho$$

$$\langle O(\phi,\pi)\rangle=\int D\phi(0)D\pi(0)O(\phi(t),\pi(t))$$



Using classical EOM

$$\ddot{\phi}-\nabla^2\phi+\frac{1}{2}m^2\phi+\frac{\lambda}{6}\phi^3=0$$



[Berges, Schmidt, Rothkopf 2008]