Thermalization of fluctuations in strongly coupled plasmas

Derek Teaney

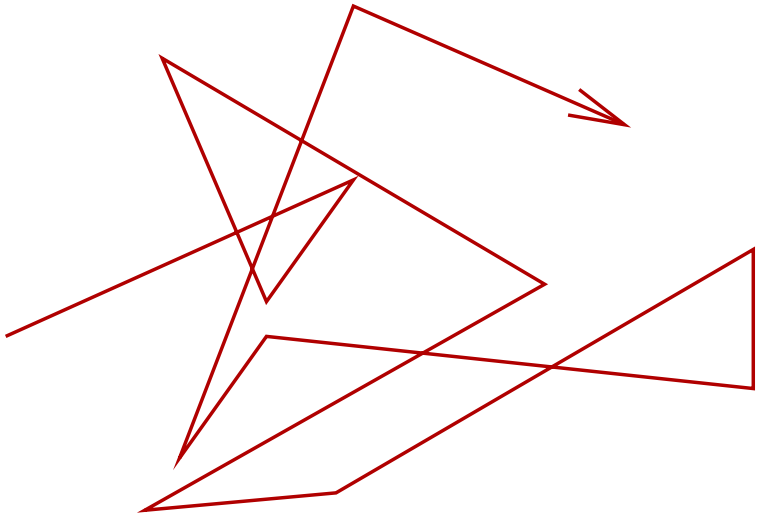
SUNY Stonybrook and RBRC Fellow



- Dam T. Son, DT; JHEP. arXiv:0901.2338
- Simon Caron-Huot, DT, Paul Chesler; PRD, arXiv:1102.1073
- Paul Chesler and DT; arXiv:1112.6196

Heavy Quarks in equilibrium Quantum Field Theory

$$M\frac{d^2\mathbf{x}}{dt^2} = \underbrace{-\eta}_{\text{Drag}}\,\dot{\mathbf{x}} + \underbrace{\xi}_{\text{Noise}}$$



"Artist's" conception of Brownian Motion

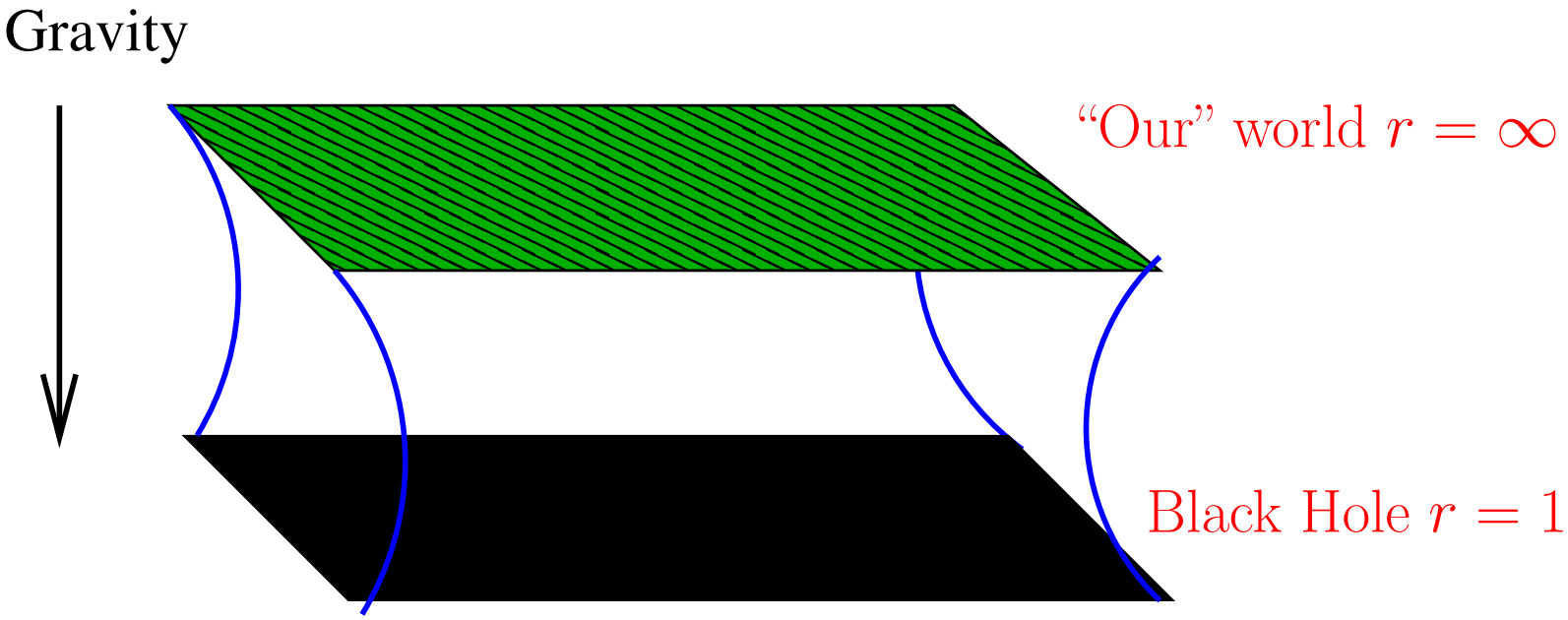
1. In equilibrium the drag and noise are balanced

$$\left\langle \xi(t)\xi(t')\right\rangle = 2T\eta\,\delta(t-t') \Leftarrow \text{Fluctuation Dissipation Theorem}$$

AdS/CFT

- Classical solutions in curved spacetime = CFT for nonzero temperature

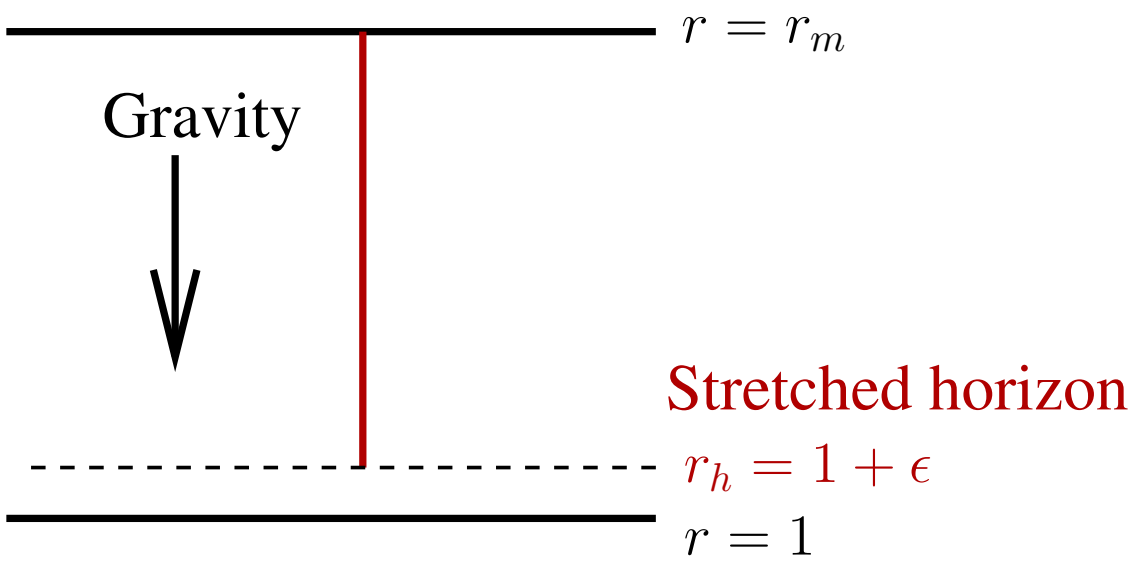
$$\mathrm{d}s^2 = (\pi T)^2 r^2 \left[-f(r)\mathrm{d}t^2 + \mathrm{d}x^2\right] + \frac{\mathrm{d}r^2}{r^2 f(r)} \qquad f(r) = 1 - \frac{1}{r^4}$$



How can a static metric be dual to equilibrium=constant fluctuations ?

Heavy Quarks in equilibrium AdS

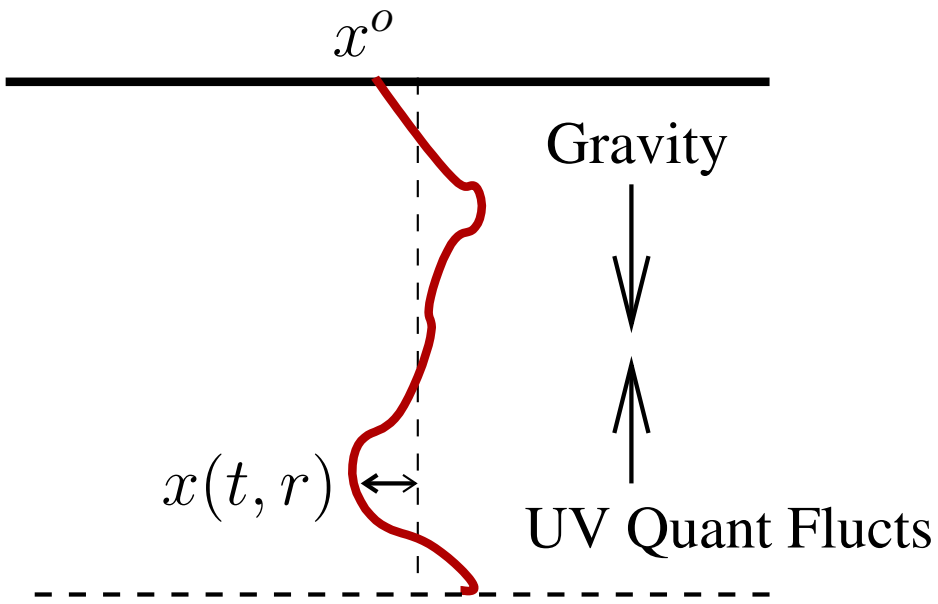
- Heavy quarks are classical strings in the 5d equilibrium AdS black hole geometry
- Solve classical string EOM and find:



Not the dual of an equilibrated quark!

Detailed Balance and Hawking Radiation:

$$M\frac{d^2x^o}{dt^2} = \underbrace{-\eta}_{\text{Drag}}\dot{x}^o + \underbrace{\xi}_{\text{Noise}}$$



Evolves to Classical

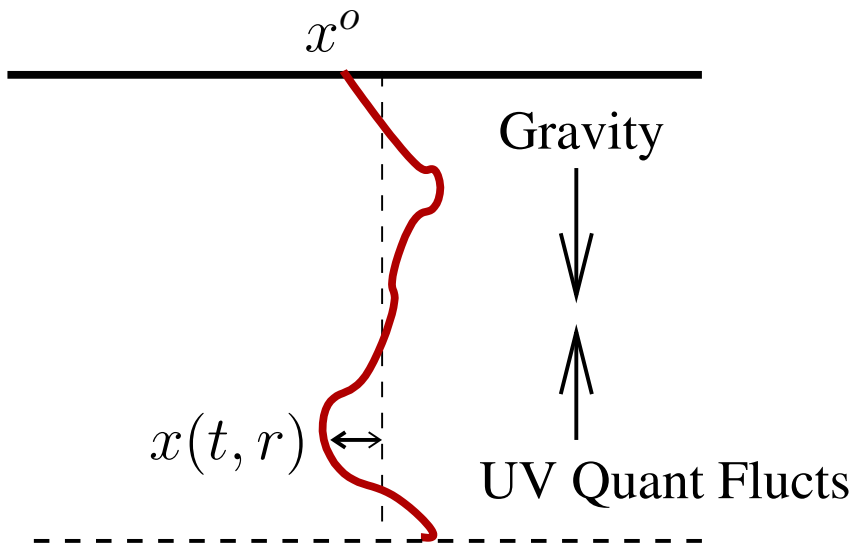
Prob Dist: (Son,DT;deBoer et al)

$$P[x,\pi_x] \propto e^{-\beta H[x,\pi_x]}$$

Goals:

1. Will show that Hawking Radiation is balanced by gravity
2. Generalize to non-equilibrium

Detailed Balance and Hawking Radiation (Technical Discussion)



1. Fluctuations:

$$G_{rr} \equiv \frac{1}{2} \langle \{ \hat{x}(t_1, r_1), \hat{x}(t_2, r_2) \} \rangle \ ,$$

2. Dissipation (Spectral Density)

$$\rho_{ra-ar} \equiv \langle [\hat{x}(t_1, r_1), \hat{x}(t_2, r_2)] \rangle \ .$$

- Equilibrium $\equiv$ Fluctuation Dissipation Theorem

$$G_{rr}(\omega, r_1, r_2) = \left(\frac{1}{2} + n_B(\omega) \right) \rho_{ra-ar}(\omega, r_1, r_2) \qquad n(\omega) \equiv \frac{1}{e^{\omega/T} - 1}$$

Formulas

- Action for string fluctuations, $h^{\mu\nu}$ = string metric

$$S_1 - S_2 = \frac{\sqrt{\lambda}}{2\pi} \int \mathrm{d}t\mathrm{d}r \, g_{xx} \left[-\sqrt{h} h^{\mu\nu} \partial_\mu x_r \partial_\nu x_a \right] ,$$

- $h^{\mu\nu}$ is the string metric

$$h_{\mu\nu} \mathrm{d}\sigma^\mu \mathrm{d}\sigma^\nu = -(\pi T)^2 r^2 f(r) dt^2 + \frac{dr^2}{f(r) r^2} ,$$

- Retarded Green Function

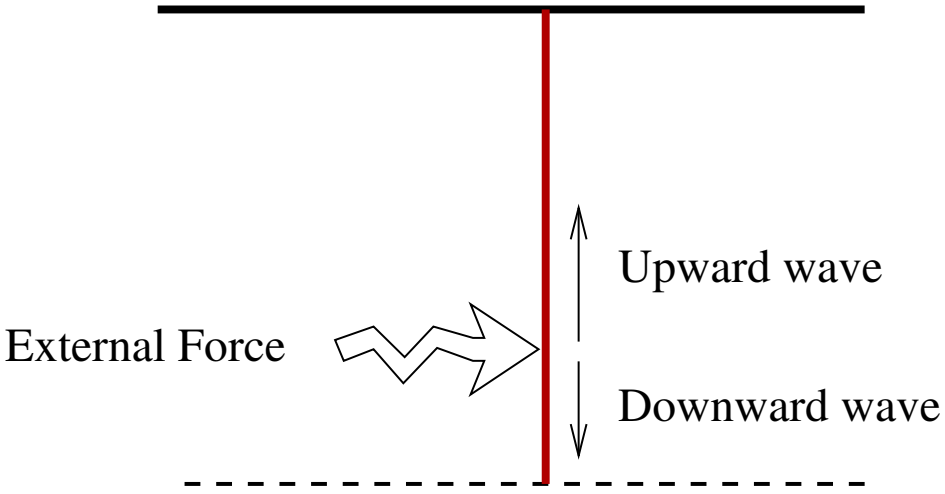
$$iG_{ra}(t_1 r_1 | t_2 r_2) \equiv \theta(t - t') \left\langle [\hat{x}(t_1, r_1), \hat{x}(t_2, r_2)] \right\rangle ,$$

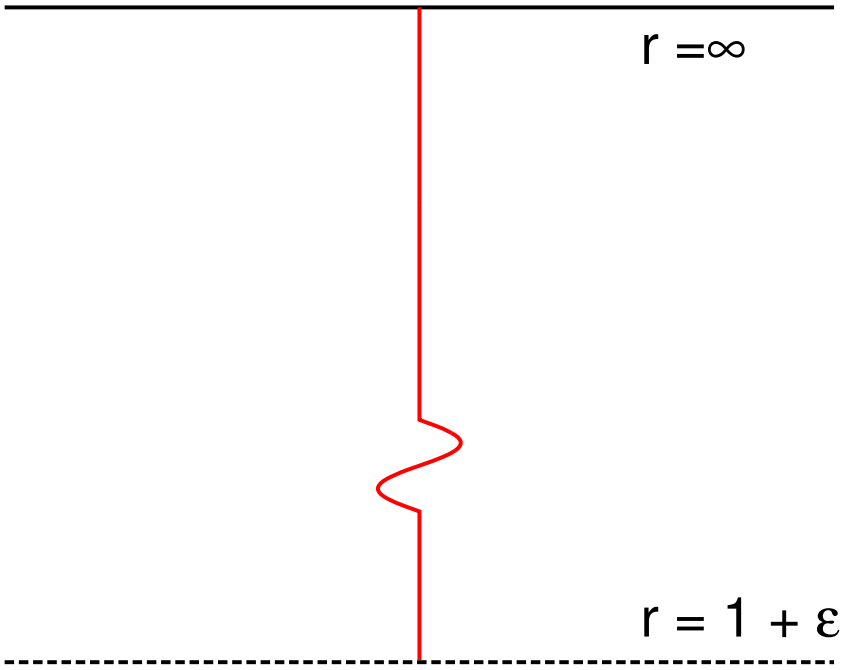
$G_{ra}(t_1 r_1 | t_2 r_2)$ is the classical response to a force at $t_2 r_2$

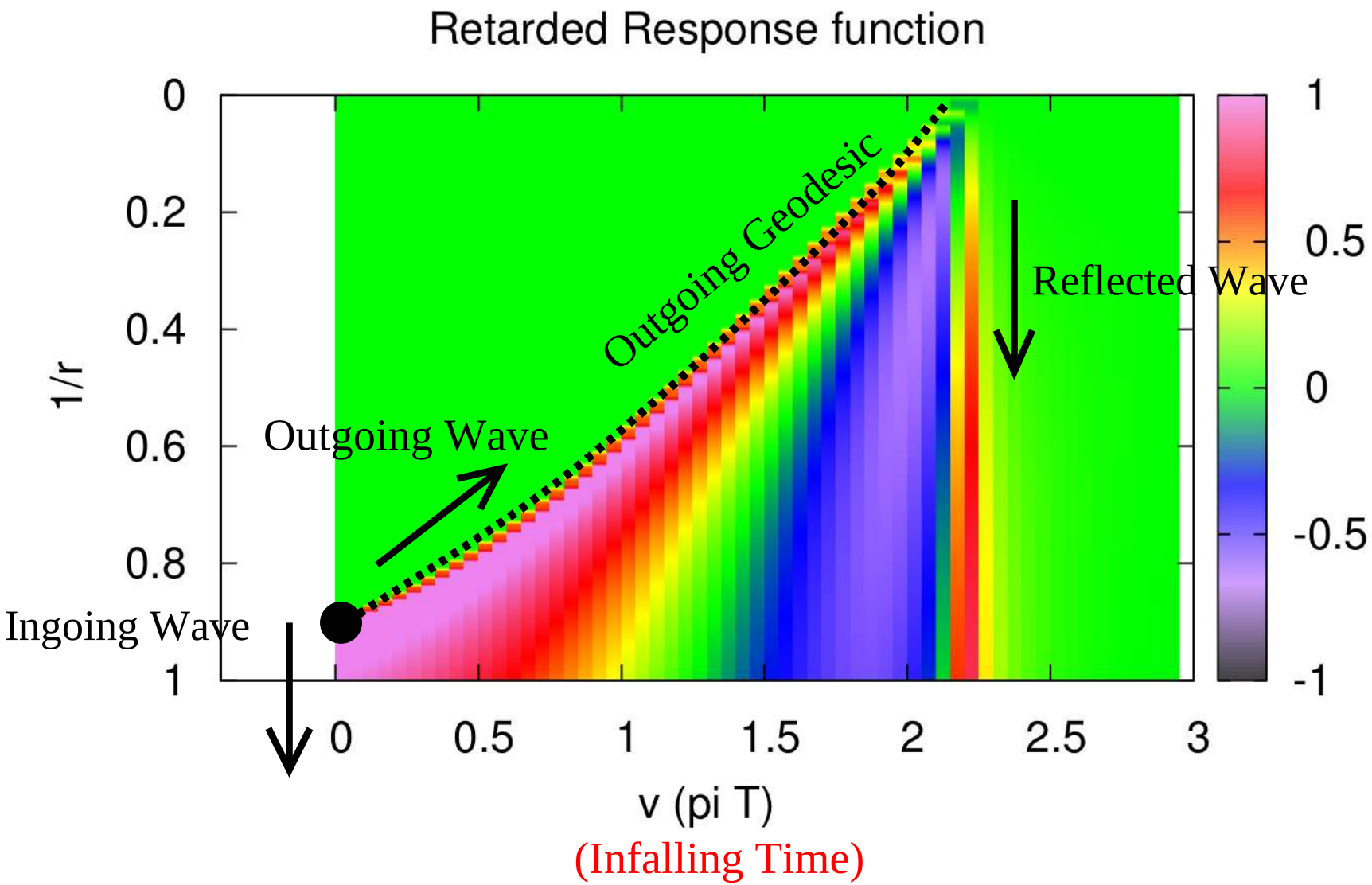
$$\frac{\sqrt{\lambda}}{2\pi} \left[\partial_\mu \, g_{xx} \sqrt{h} h^{\mu\nu} \partial_\nu \right] G_{ra}(t_1 r_1 | t_2 r_2) = \delta(t_1 - t_2) \delta(r_1 - r_2) ,$$

The classical Green Function or response to a force:

$$\frac{\sqrt{\lambda}}{2\pi}\left[\partial_\mu\, g_{xx}\sqrt{h}h^{\mu\nu}\partial_\nu\right] G = \mathcal{F}\,\delta(t_1-t_2)\delta(r_1-r_2)\,,$$



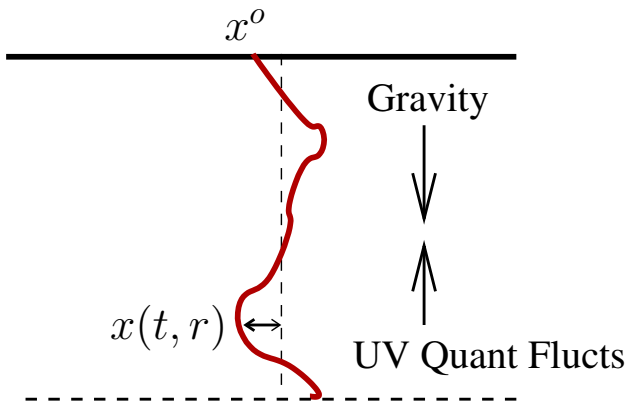




$$v = t - \frac{1}{2\pi T}\left[\tan^{-1}(r) + \tanh^{-1}(r)\right]$$

v =Eddington time

Statistical Fluctuations



$$G_{rr} = \frac{1}{2}\left\langle\{x(t_1,r_1), x(t_2,r_2)\}\right\rangle$$

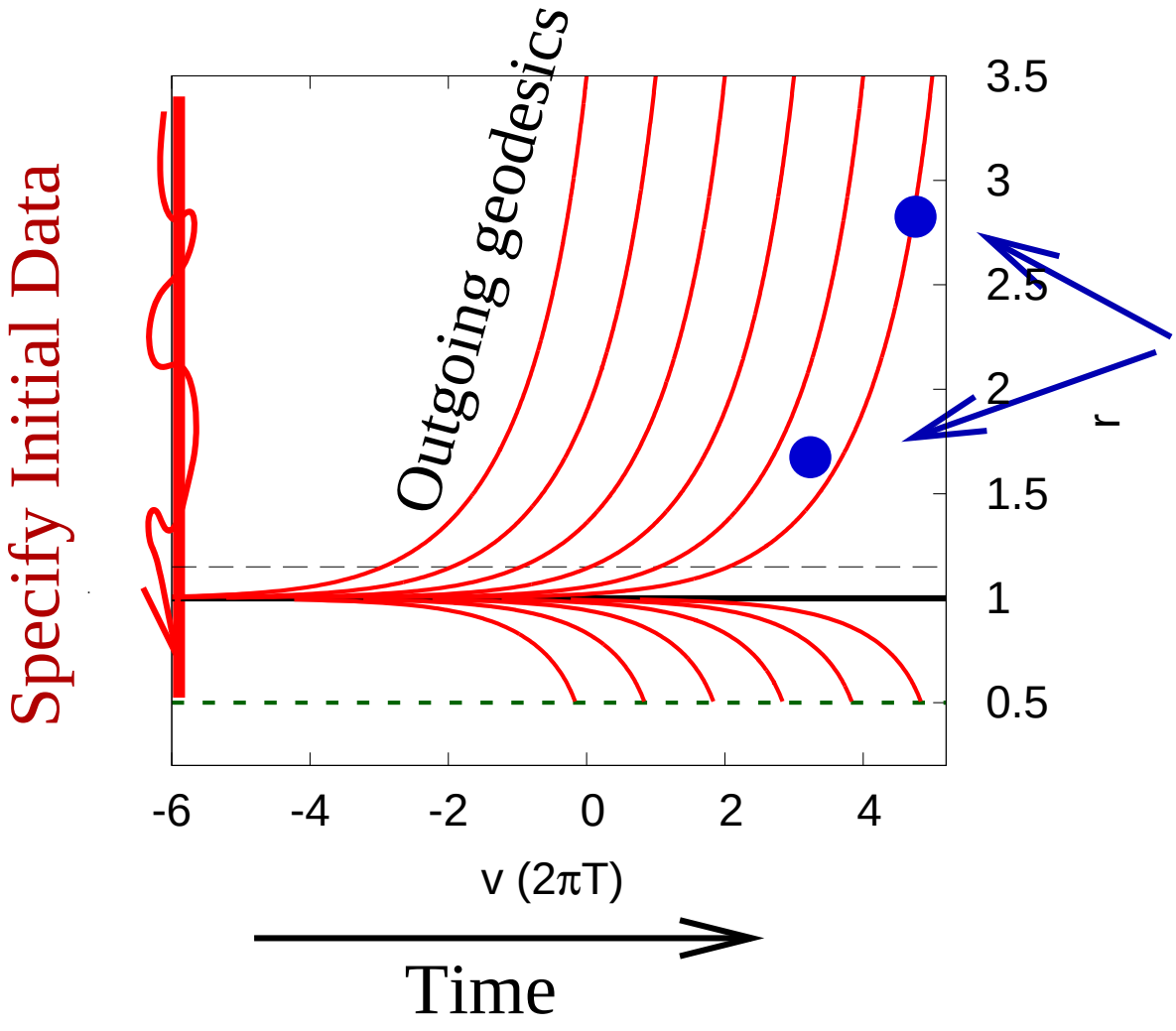
- The statistical correlator obeys the homogeneous EOM

$$\frac{\sqrt{\lambda}}{2\pi}\left[\partial_\mu\, g_{xx}\sqrt{h}h^{\mu\nu}\partial_\nu\right] G_{rr}(t_1 r_1|t_2 r_2) = 0$$

- So:

1. Specify the correlations (or density matrix) in the past
2. Final state fluctuations are correlated only through initial conditions

Correlations through Initial conditions

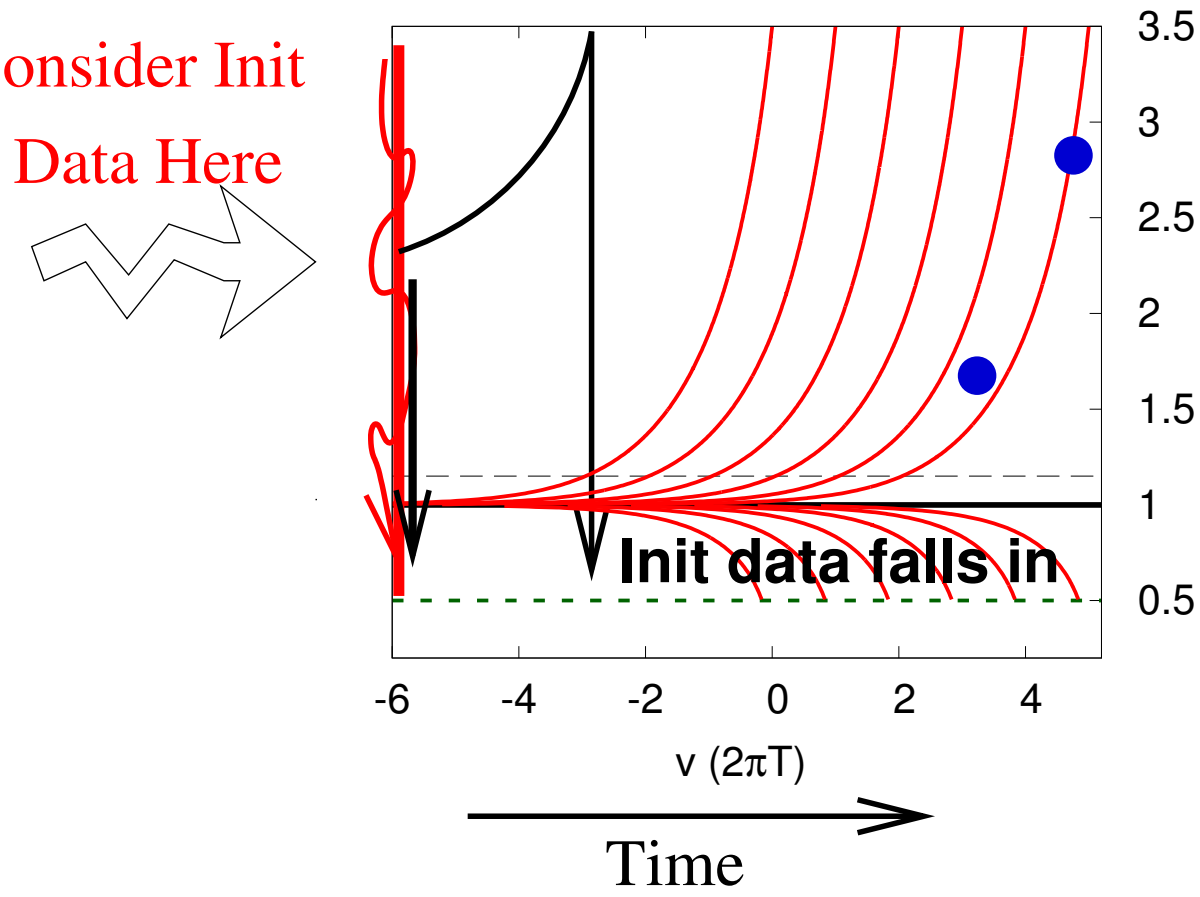


Correlated through

Initial conditions

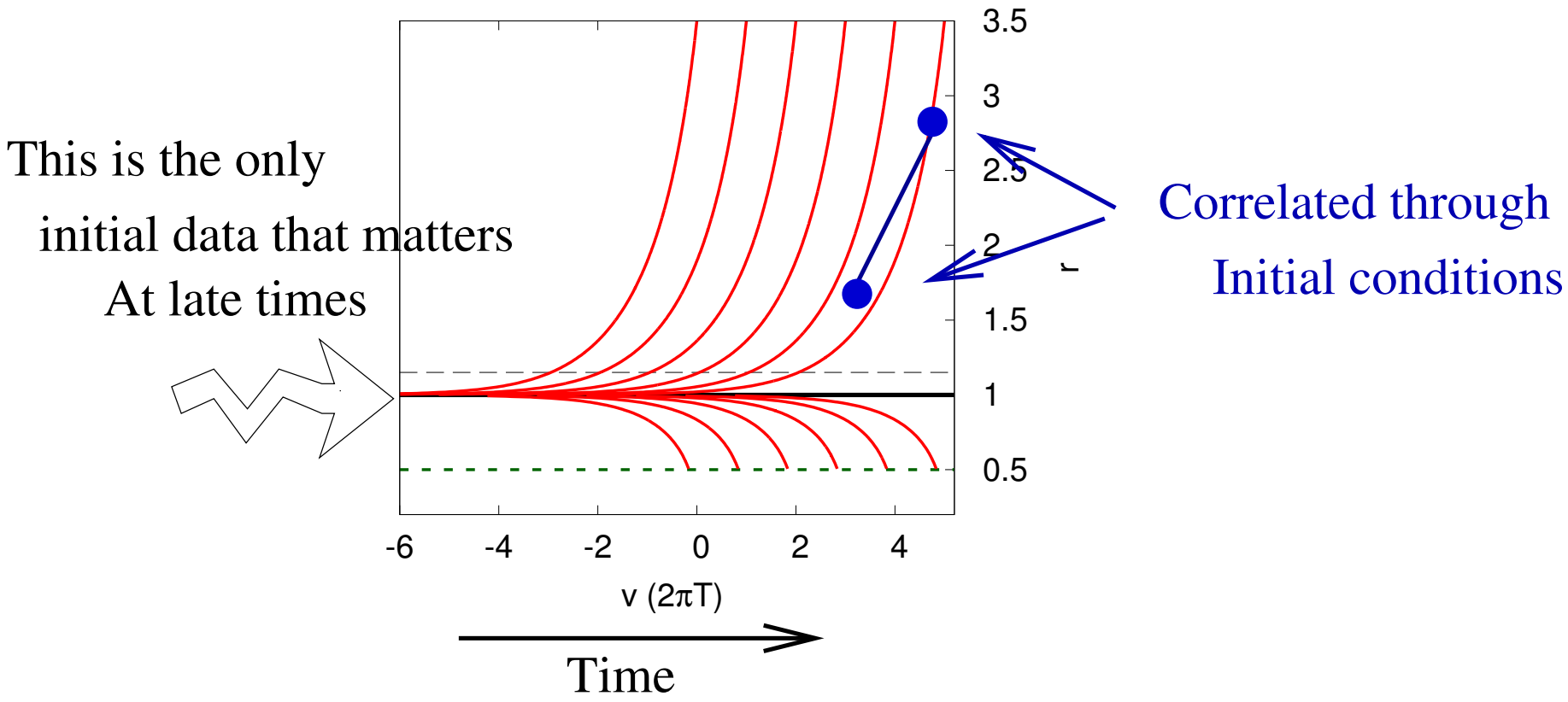
Correlations through Initial conditions

Consider Init Data Here



Points uncorrelated by this Init data

Correlations through Initial conditions



1. Final correlation come from correlated initial data very near horizon
   - Short Wavelength
2. Initial data is inflated by near horizon geometry

Initial Data from Quantum Fluctuations

1. Initial data is determined at short distance = Flat Space Physics

2. Scalar Field in 1+1D vacuum flat space

$$\frac{1}{2}\left\langle\{\phi(X_1),\phi(X_2)\}\right\rangle = -\frac{1}{4\pi K}\log\left|\mu\,\eta_{\mu\nu}\Delta X^\mu \Delta X^\nu\right|$$

K=norm of action

3. String flucts in near horizon geometry

$$S^{\rm near-horizon} = \eta\int {\rm d}t{\rm d}r\,\left[-\frac{1}{2}\sqrt{h}h^{\mu\nu}\partial_\mu x\partial_\nu x\right]$$

$\eta =$ Drag Coefficient

The near horizon initial condition is:

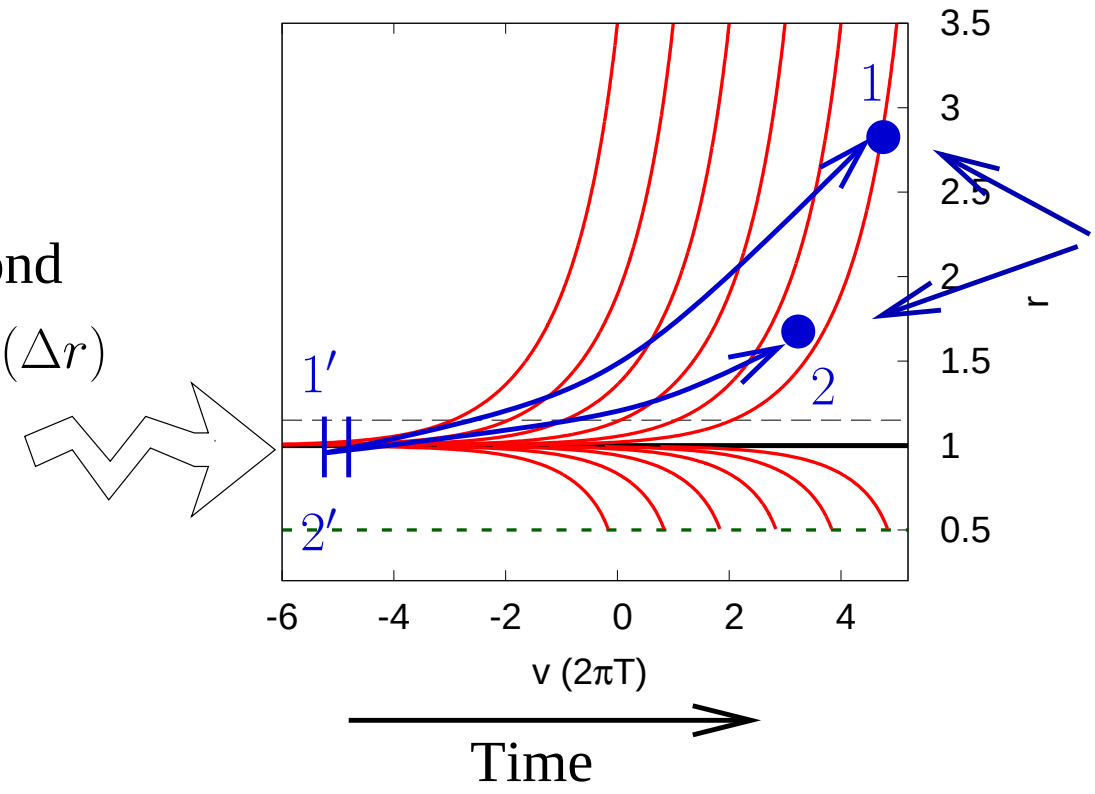
$$G_{rr}(v_1 r_1|v_2 r_2) \to -\frac{1}{4\pi\eta}\log\left|\mu\,\overbrace{2\Delta v\,\Delta r}^{\text{local }\Delta s^2}\right|$$

Summary: Specify IC and Solve Equations of Motion

$$\frac{\sqrt{\lambda}}{2\pi}\left[\partial_\mu\, g_{xx}\sqrt{h}h^{\mu\nu}\partial_\nu\right] G_{rr}(t_1 r_1|t_2 r_2) = 0$$

Init. Cond

$\propto \log(\Delta r)$



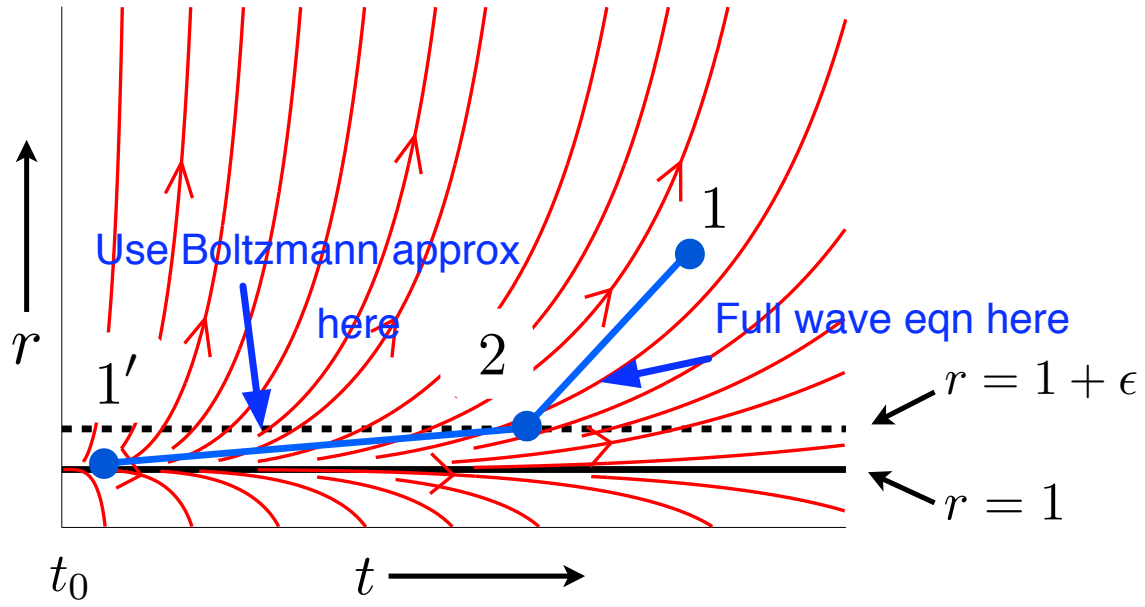
Correlated via log Init Cond

$$G_{rr}(1|2) = \left[\frac{\sqrt{\lambda}}{2\pi}\int \mathrm{d}r_1'\; g_{xx}\sqrt{h}h^{tt}(r_1')\, G_{ra}(1|1')\overleftrightarrow{\partial}_{t_1'}\right]$$

Like Harmonic Oscllator

$$\times\left[\frac{\sqrt{\lambda}}{2\pi}\int \mathrm{d}r_2'\; g_{xx}\sqrt{h}h^{tt}(r_2')\, G_{ra}(2|2')\overleftrightarrow{\partial}_{t_2'}\right] G_{rr}(1'|2')\,,$$

From initial data to final correlations in two steps:



$$G_{ra}(1|1') = \int \mathrm{d}t_2 \overbrace{G_{ra}(1|2)}^{\text{in bulk}} \Big[\eta\sqrt{h}h^{rr}(r_2)\partial_{r_2}\Big]_{r_2=1+\epsilon} \overbrace{g_{ra}(2|1')}^{\text{in strip}}$$

(a) From horizon to stretched horizon – Waves are very short wavelength

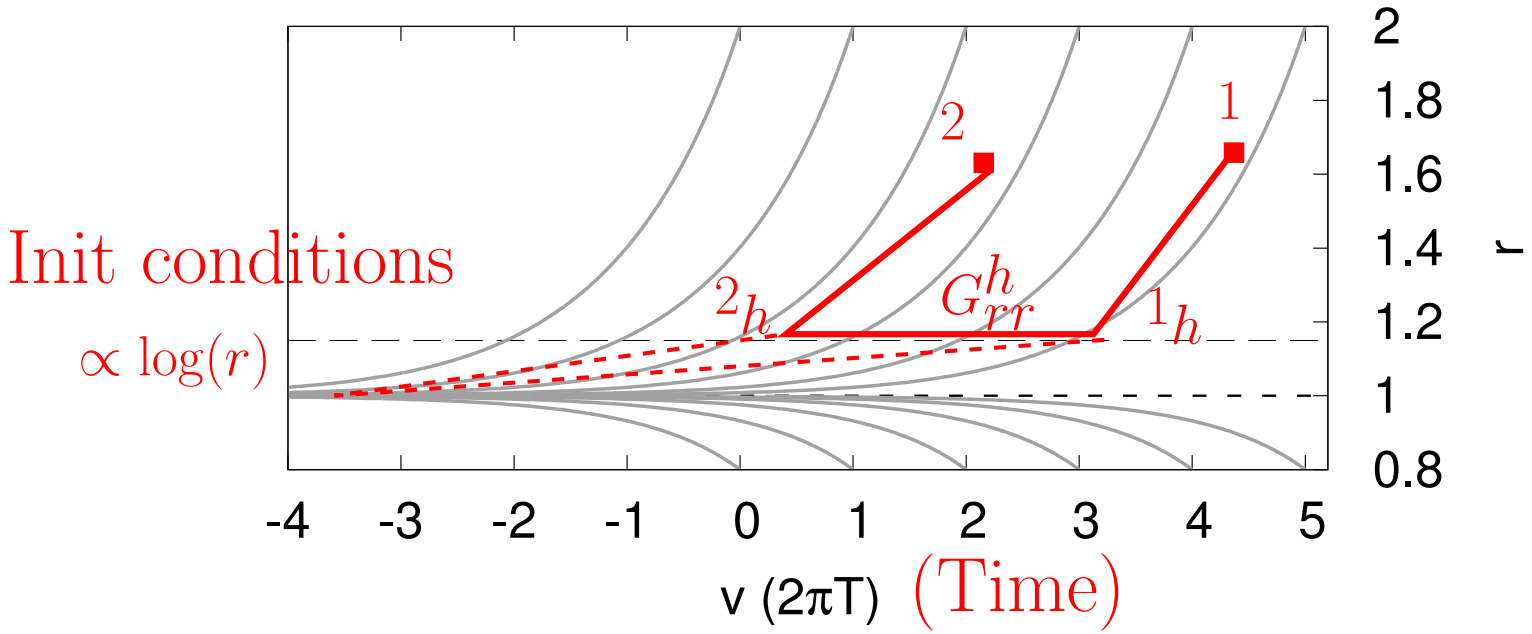
- Will use free Boltzmann approximation (geodesic/WKB/eikonal approx)

(b) The stretched horizon to boundary – Waves are longer wavelength

- Will use full wave equation

Fluctuations from Equations of Motion

$$\underbrace{G_{rr}(1|2)}_{\text{bulk flucts}} = \int \mathrm{d}t_{1h}\mathrm{d}t_{2h}\ \underbrace{G_R(1|1_h)\,G_R(2|2_h)}_{\text{outgoing Green fcns}}\ \underbrace{G^h_{rr}(1_h|2_h)}_{\text{horizon flucts}}\,,$$

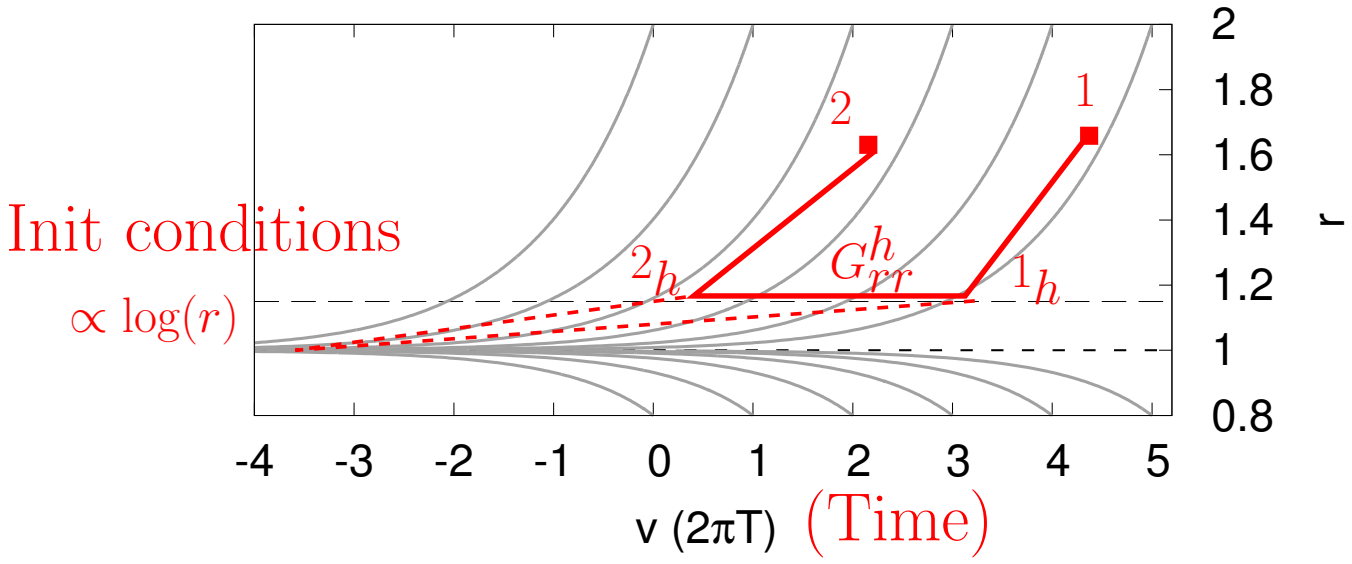


The horizon correlator is found from the horizon limit of correlation fcn in strip

$$G^h_{rr}(t_1|t_2) = \text{Blow-up of initial data} \propto \log(r)$$

$$= -\frac{\eta}{\pi}\partial_{t_1}\partial_{t_2}\log|1-e^{-2\pi T(t_1-t_2)}|\,.$$

The horizon fluctuations and the Lyapunov exponent



1. Thermal looking:

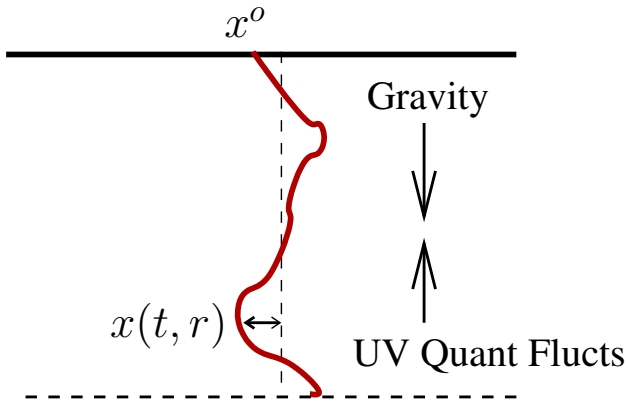
$$G^h_{rr}(\omega) = \text{Fourier-Trans of } -\frac{\eta}{\pi}\partial_{t_1}\partial_{t_2}\log\left|1-e^{-2\pi T(t_1-t_2)}\right|$$

$$= \left(\tfrac{1}{2}+n(\omega)\right)2\omega\eta \qquad\qquad n(\omega)\equiv\frac{1}{e^{\omega/T}-1}$$

2. Temperature $\propto$ inflation rate

$$2\pi T = \text{Lyapunov exponent of diverging geodesics}$$

Dissipation - Spectral Density



$$\rho_{ra-ar} = \langle [\hat{x}(t_1, r_1), \hat{x}(t_2, r_2)] \rangle$$

- The spectral density also obeys the EOM

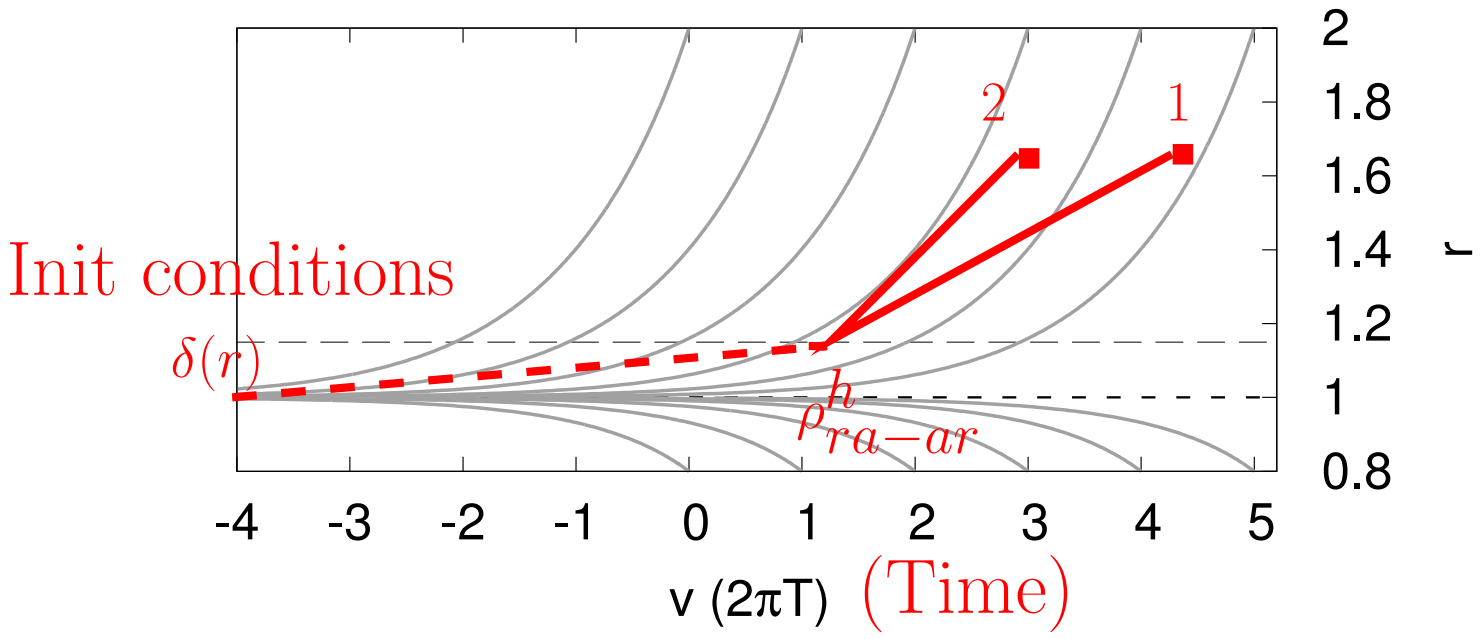
$$\frac{\sqrt{\lambda}}{2\pi} \left[\partial_\mu \, g_{xx} \sqrt{h} h^{\mu\nu} \partial_\nu \right] \rho_{ra-ar}(t_1 r_1 | t_2 r_2) = 0$$

- But initial conditions are given by the canonical commutation relations

$$\eta \sqrt{h} h^{tt}(r_1) \lim_{t_2 \to t_1} \partial_{t_1} \rho_{ra-ar}(t_1 r_1 | t_2 r_2) = i\delta(r_1 - r_2) \, .$$

Spectral Density

$$\underbrace{\rho_{ra-ar}(1|2)}_{\text{bulk spectral fcn}} = \int \mathrm{d}t_{1h}\mathrm{d}t_{2h}\ \underbrace{G_R(1|1_h)\,G_R(2|2_h)}_{\text{outgoing Green fcns}}\ \underbrace{\rho^h_{ra-ar}(1_h|2_h)}_{\text{horizon spectral fcn}}\ ,$$



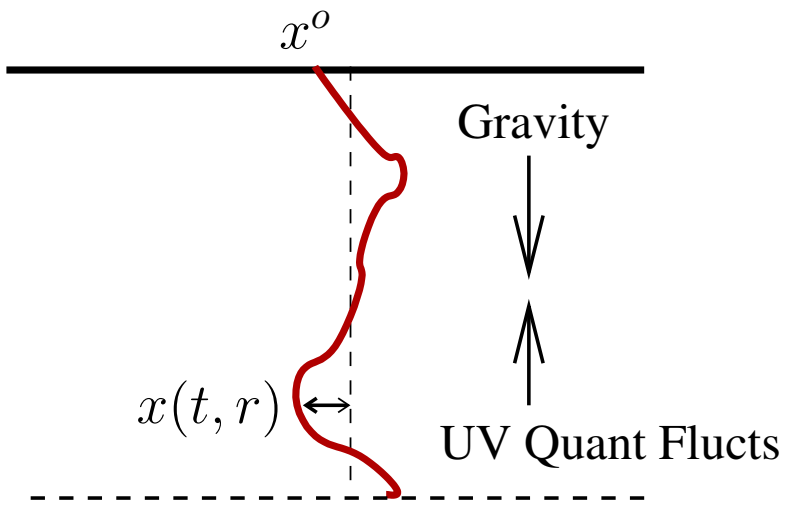
Where the horizon spectral density

$$\rho^h_{ra-ar}(t_1,t_2) = \text{local due to canonical commutation relations}$$

$$= 2\eta\left[-i\delta'(t_1-t_2)\right] \qquad (2\omega\eta \text{ in Fourier space})$$

Detailed Balance

$$G_{rr}(\omega, r_1, r_2) = \left(\tfrac{1}{2} + n(\omega)\right) \rho(\omega, r_1, r_2)$$



1. Fluctuations (Anti-commutator)

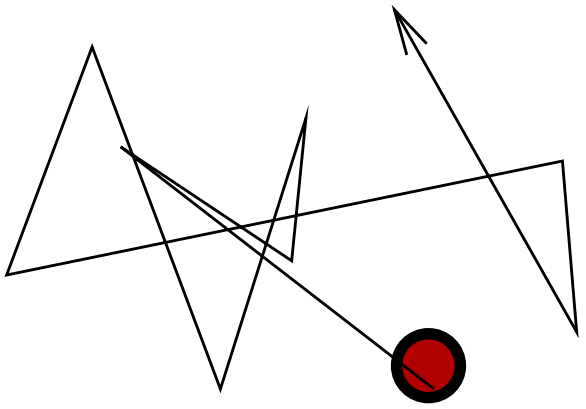
$$\underbrace{G_{rr}(\omega, r_1, r_2)}_{\text{bulk flucts}} = \underbrace{G_R(\omega, r_1|r_h)\, G_R(-\omega, r_2|r_h)}_{\text{outgoing Green fcns}} \underbrace{\left(\tfrac{1}{2} + n(\omega)\right) 2\omega\eta}_{\text{Horizon-flucts}}$$

2. Dissipation: (Commutator)

$$\underbrace{\rho_{ra-ar}(\omega, r_1, r_2)}_{\text{bulk spec dense}} = \underbrace{G_R(\omega, r_1|r_h)\, G_R(-\omega, r_2|r_h)}_{\text{outgoing Green fcns}} \quad \underbrace{2\omega\eta}_{\text{Horizon spec dense}}$$

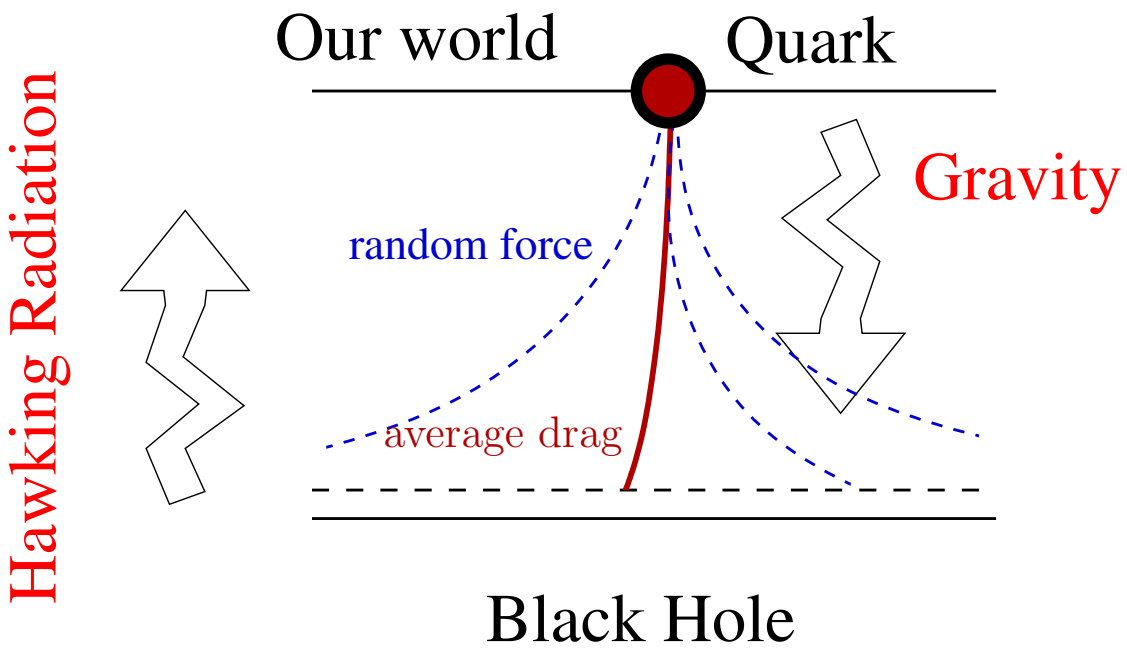
The Picture

Brownian Quark



$$M\frac{dv}{dt} = -\eta v + \xi(t)$$

Stochastic String Pulling on Quark



5D equilibrium is a competition between dissipative gravity and hawking radiation:

classical probability $\propto e^{-\beta H[x,\pi_x]}$

Reproduces the Brownian motion of a Heqvy Quark

Non-equilibrium

Non-equilibrium: (Chesler-Yaffe)



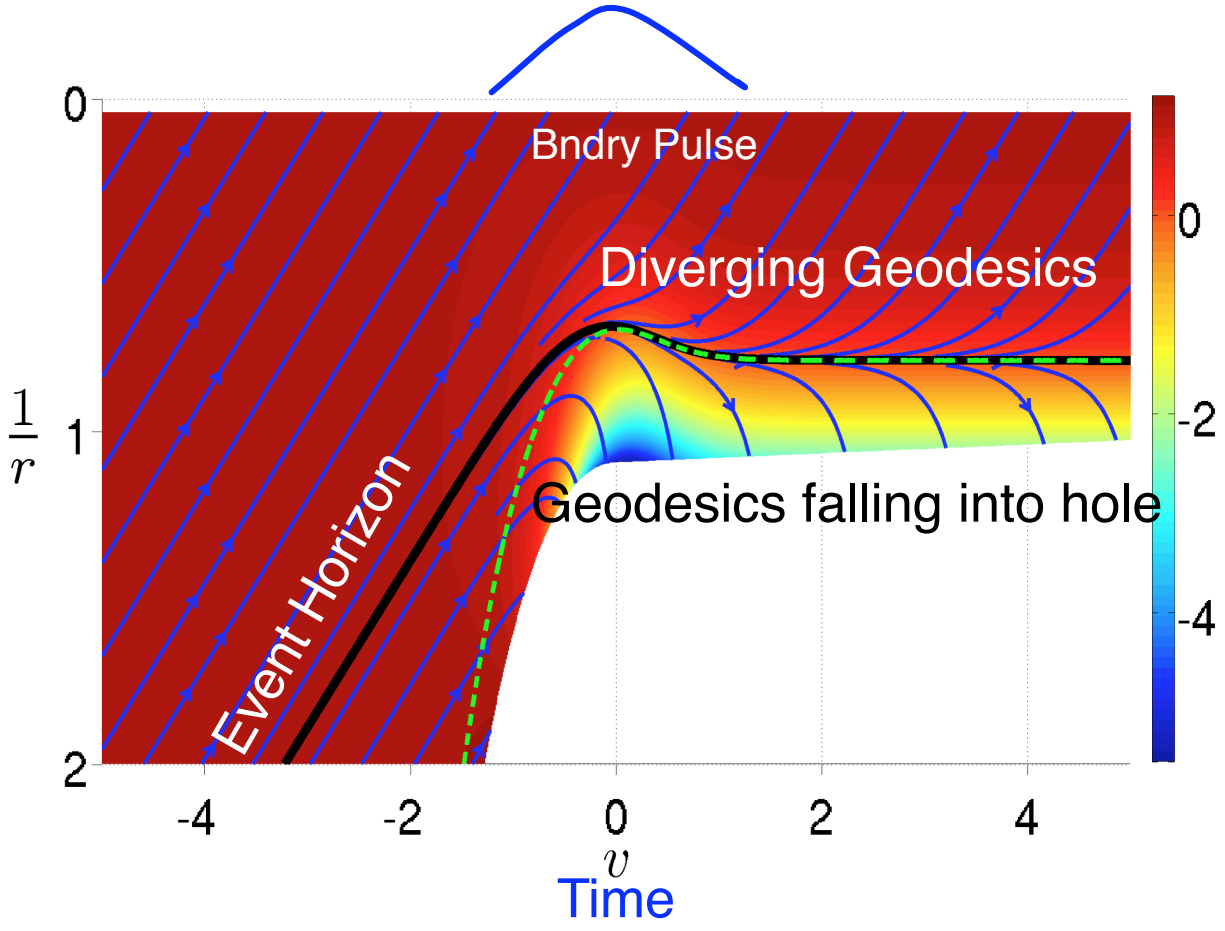


- God turns on a strong gravitational pulse in “Our” world

$$ds^2 = -dt^2 + e^{B_o(t)} d\mathbf{x}_\perp^2 + e^{-2B_o(t)} dx_\parallel^2$$



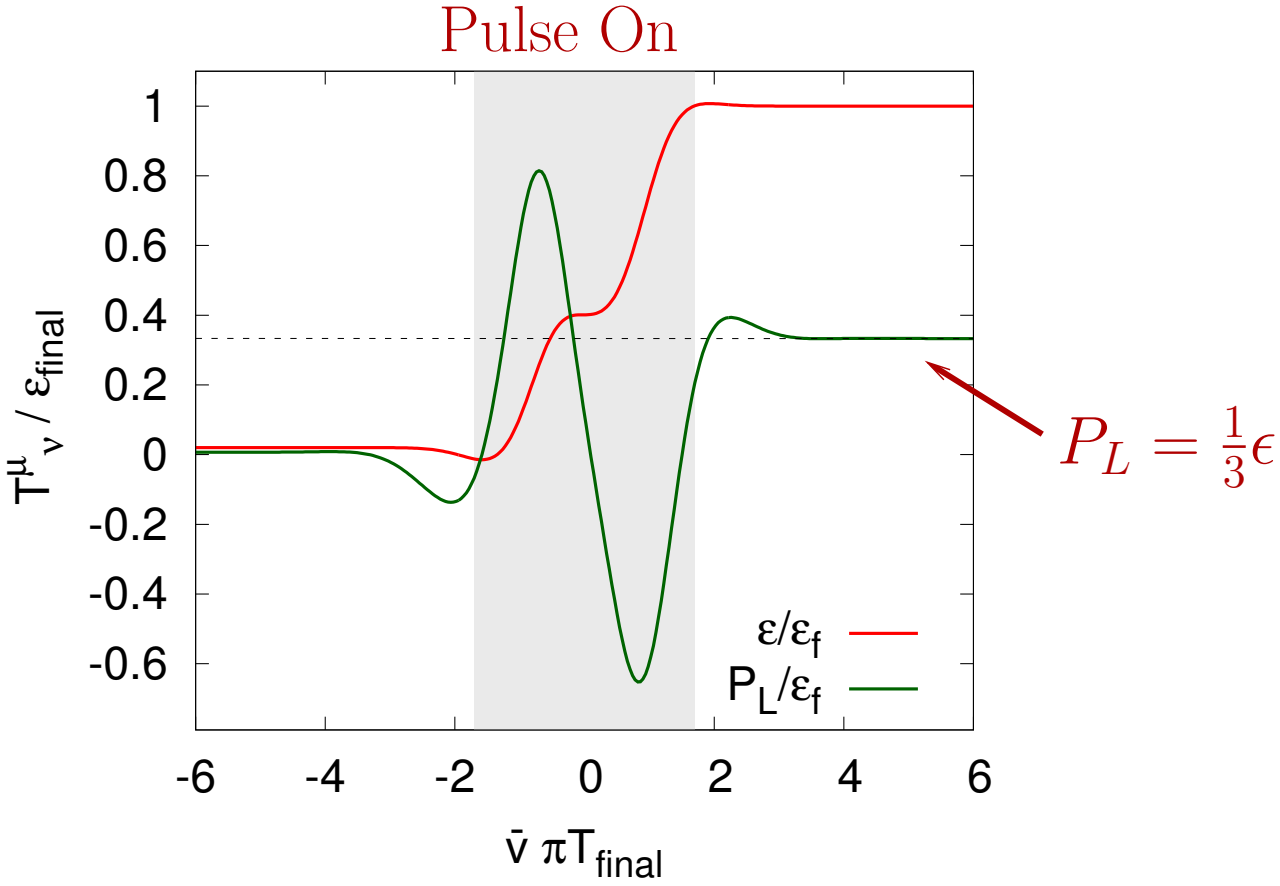
1. Corresponds to non-equilibrium geometry with BH formation in AdS_5



The boundary stress tensor

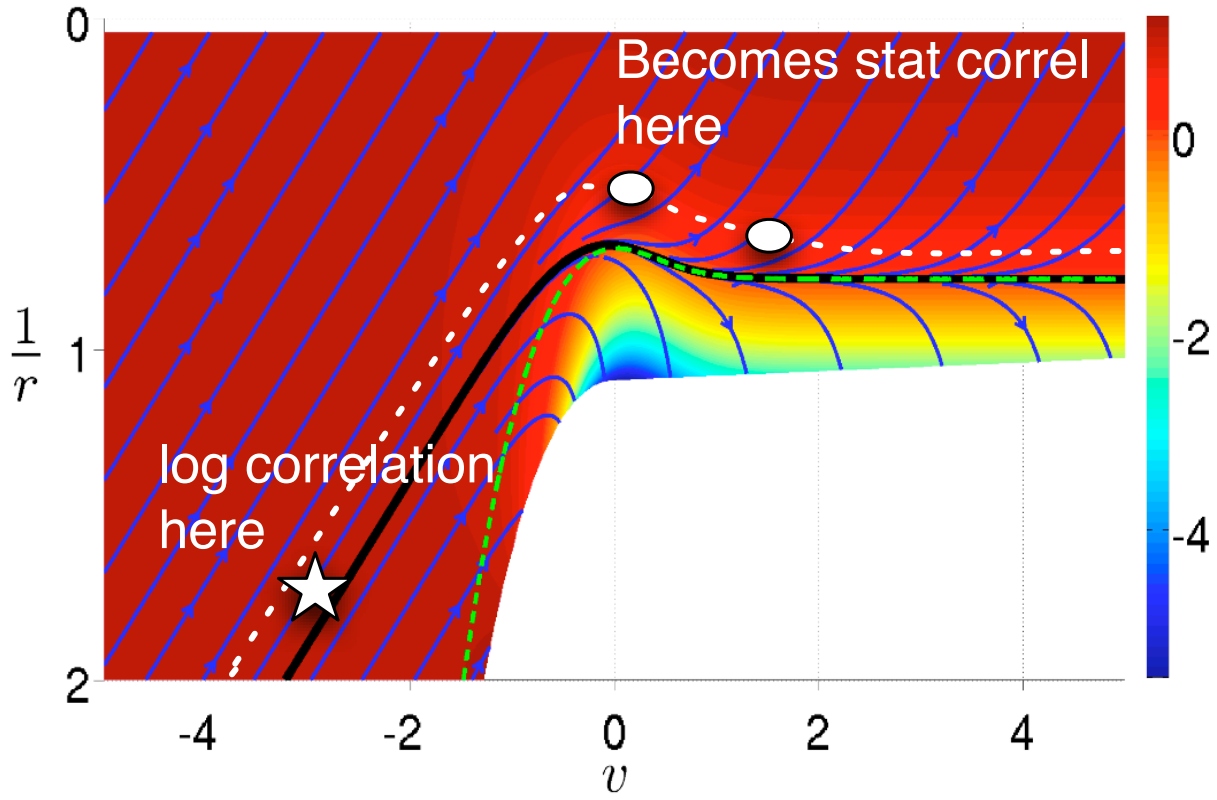
- Paul and Larry turn on a strong gravitational pulse in “Our” world

$$ds^2 = -dt^2 + e^{B_o(t)} d\mathbf{x}_\perp^2 + e^{-2B_o(t)} dx_\parallel^2$$



$$T^\mu_\nu = \mathrm{diag}(\mathcal{E}, \mathcal{P}_T, \mathcal{P}_T, \mathcal{P}_L)$$

Fluctuations in non-equilibrium



- Surface Properties – on event horizon

$$\underbrace{2\pi T_{\rm eff}(v)}_{\text{Lyapunov exponent}} \equiv \overbrace{\frac{1}{2}\frac{\partial A(r,v)}{\partial r}}^{\text{Metric-coeff}}\Bigg|_{r=r_h(v)} \qquad \propto \text{extrinsic curvature}$$

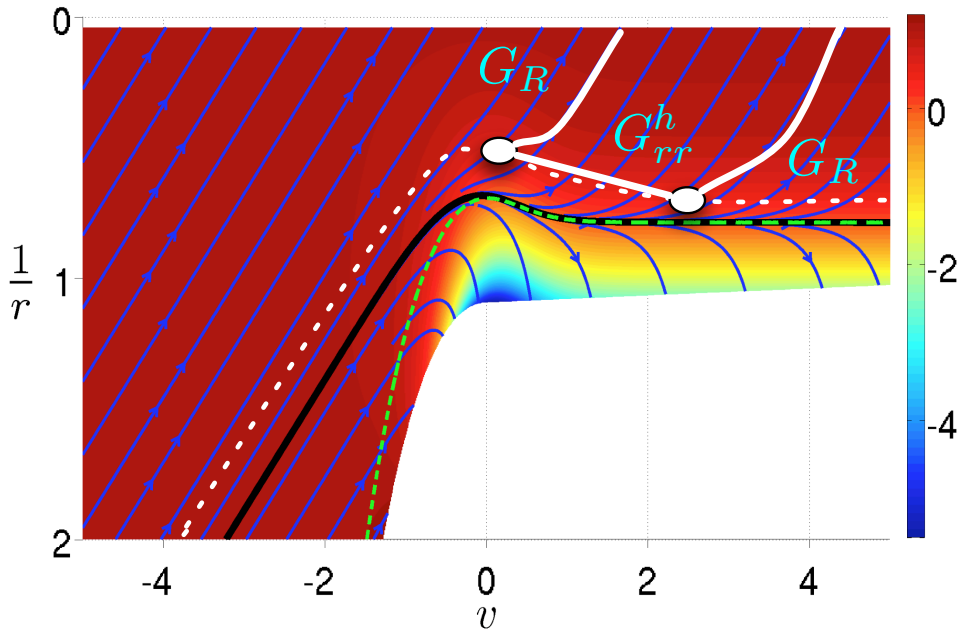
Result:

- General form of near horizon fluctuations in non-equilibrium

$$G^h_{rr}(v_1|v_2) = -\frac{\sqrt{\eta(v_1)\eta(v_2)}}{\pi}\partial_{v_1}\partial_{v_2}\log\left|1-e^{-\int_{v_1}^{v_2}2\pi T_{\rm eff}(v')dv'}\right| .$$

- Can map the near horizon fluctuations up to boundary

$$G_{rr}(1|2) = \int \mathrm{d}v_{1h}\mathrm{d}v_{2h}\ \underbrace{G_R(1|1_h)\,G_R(2|2_h)}_{\text{outgoing Green Fcns}}\ \underbrace{G^h_{rr}(1_h|2_h)}_{\text{horizon flucts}} .$$



What to compute: Emission and Absorption Rates

- Emission: In equilibrium the emission rate of dilatons is

$$(2\pi)^3 2k \frac{d\Gamma}{d^3k} = G^<(K) \qquad G^<(K) = \langle J(0)J(K)\rangle$$

- Absorption: The absorption rate of Dilatons is

$$(2\pi)^3 2k \frac{d\Gamma}{d^3k} = G^>(K) \qquad G^>(K) = \langle J(K)J(0)\rangle$$

- FDT: The Fluctuation Dissipation Relation reads

$$\underbrace{G^<(K)}_{\text{emission}} = e^{-\omega\beta} \underbrace{G^>(K)}_{\text{absorption}} \qquad \beta \equiv \frac{1}{T}$$

We will compute the emission and absorption rates and check for detailed balance

A non-equilibrium definition of the Emission and Absorption Rates

Want to know the rate to emit and absorb a *high* frequency ω at time t

1. Wigner Transforms – perfect frequency resolution, but no time resolution

$$G^{<}(\bar{t},\omega) = \int_{-\infty}^{\infty} \mathrm{d}\Delta t e^{+i\omega\Delta t} \left\langle J(\bar{t}-\Delta t)J(\bar{t}+\Delta t)\right\rangle$$

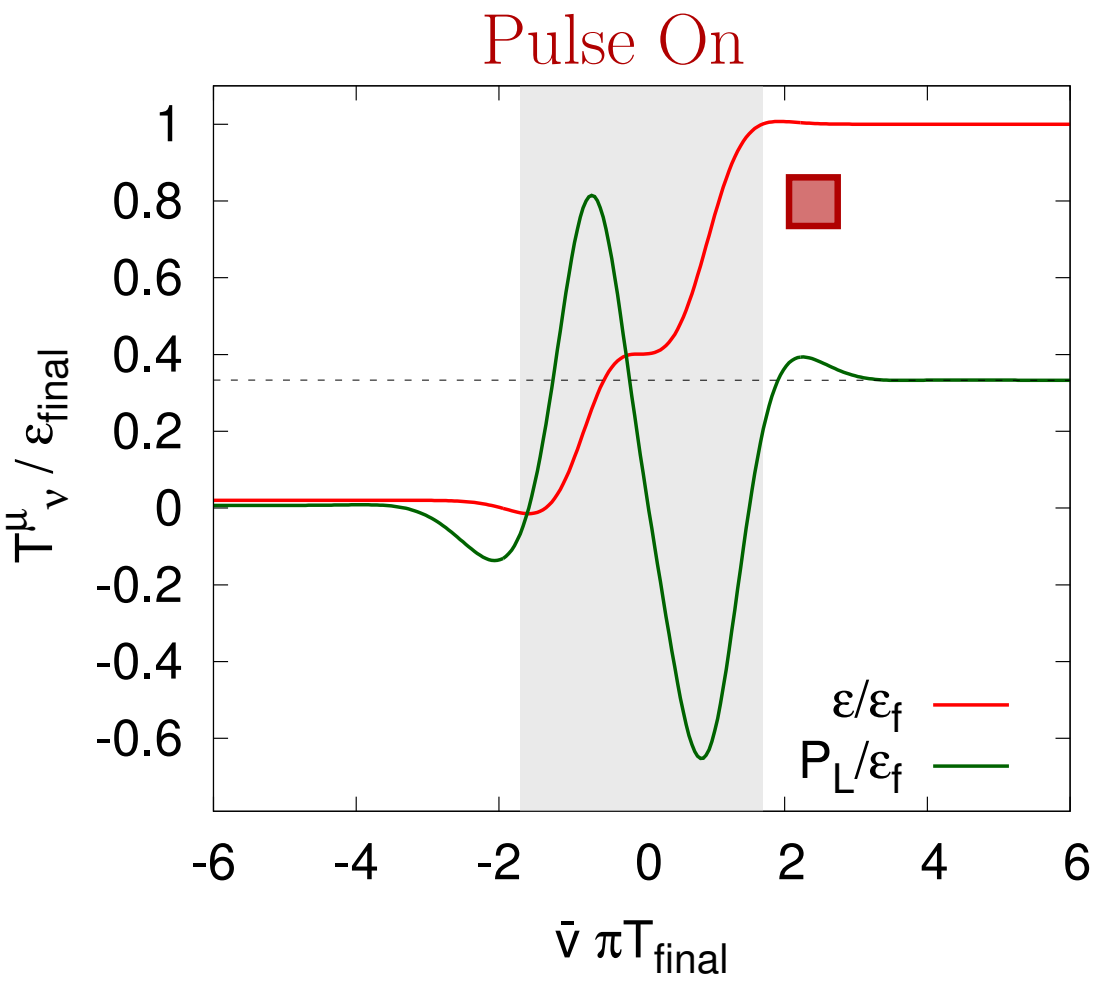
2. Gabor Transform – Wigner smeared with a minimum uncertainty wave packet

$$\underbrace{\bar{G}^{<}(\bar{t}_o,\bar{\omega}_o)}_{\text{Gabor}} = \int \frac{dtd\omega}{2\pi} \underbrace{2e^{-(\omega-\omega_o)^2\sigma^2} e^{-(\bar{t}-\bar{t}_o)^2/\sigma^2}}_{\text{minimum wave packet}} \underbrace{G^{<}(\bar{t},\omega)}_{\text{Wigner Trans}}$$

3. In equilibrium Gabor transforms satisfy a modified FDT

$$\underbrace{\bar{G}^{<}(\bar{t}_o,\bar{\omega}_o)}_{\text{emission}} = e^{-\omega_o/T} \underbrace{\left[e^{\beta^2/4\sigma^2}\bar{G}^{>}(\bar{t}_o,\bar{\omega}_o-\beta/2\sigma^2)\right]}_{\text{absorption}}$$

We will test this equality to monitor equilibrium



- Temporal Resolution

$$\sigma_v = \frac{1}{\sqrt{2}} \simeq 0.7$$

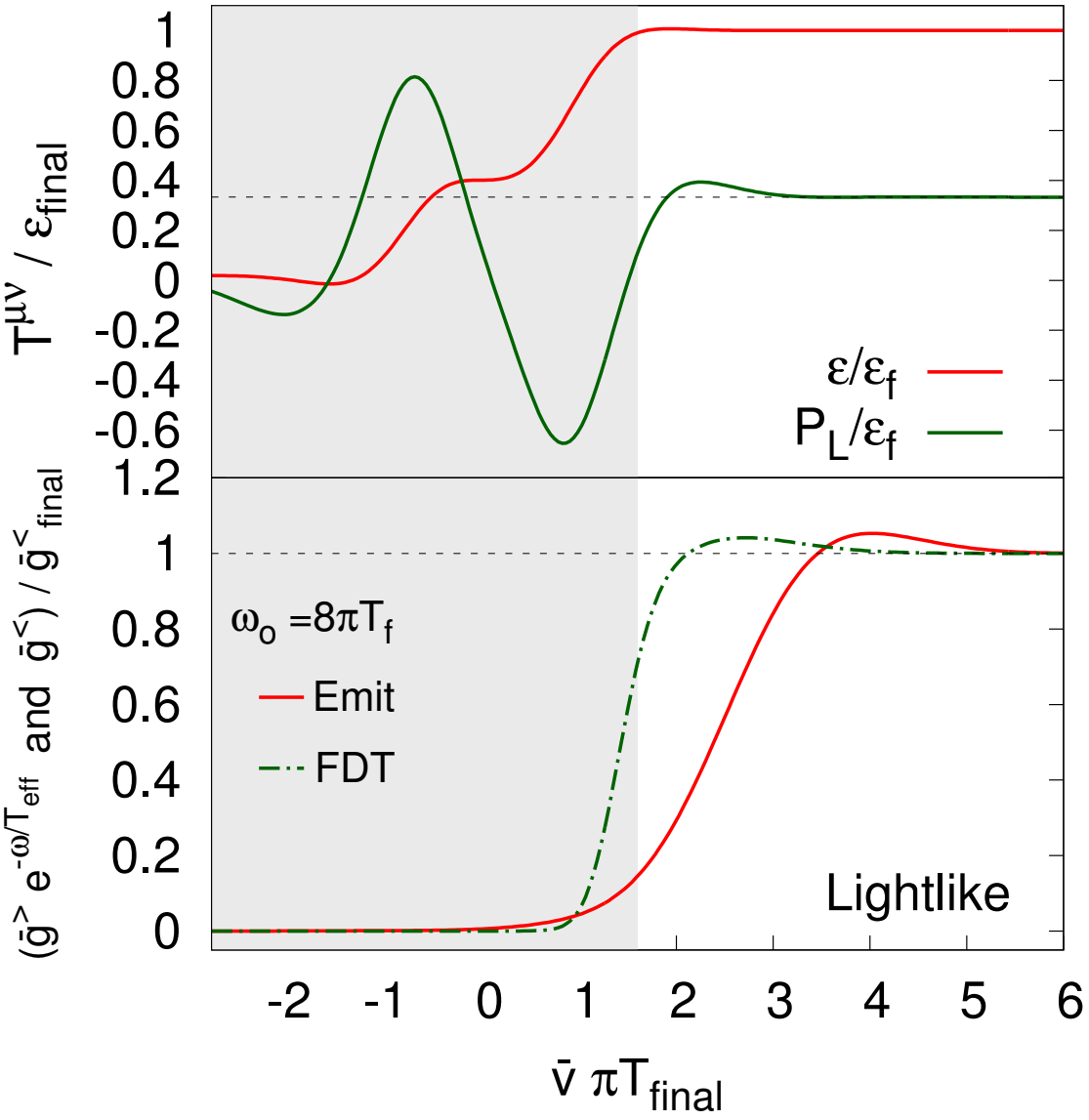
- Frequency Resolution

$$\frac{\sigma_\omega}{\omega} \simeq \frac{1/\sqrt{2}}{8} \simeq 10\%$$

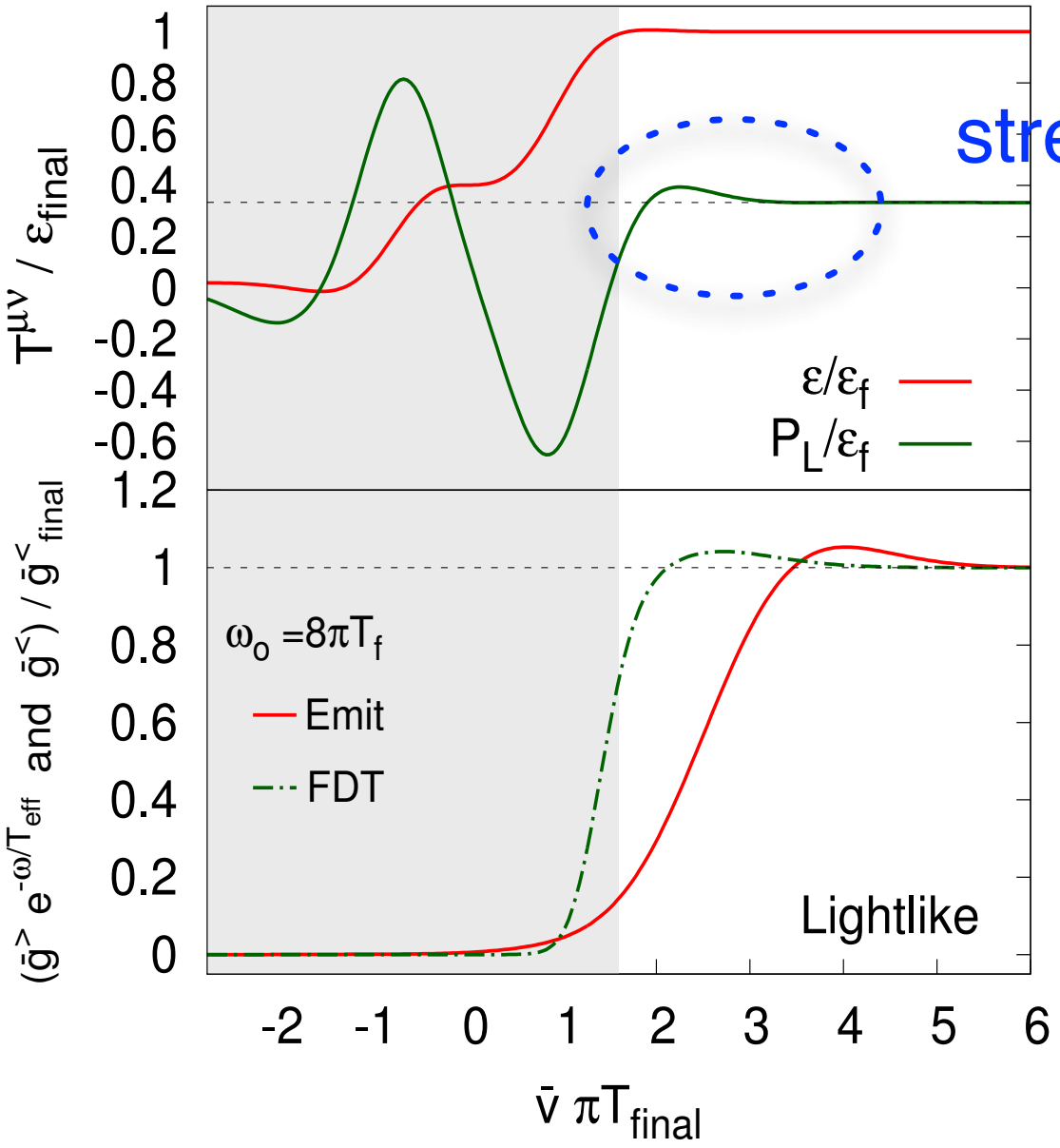
- Effective temperature

$$T_{\rm eff}(v) \propto \mathcal{E}^{1/4}$$

Lightlike emission rate $\omega_o = 8\pi T_f$

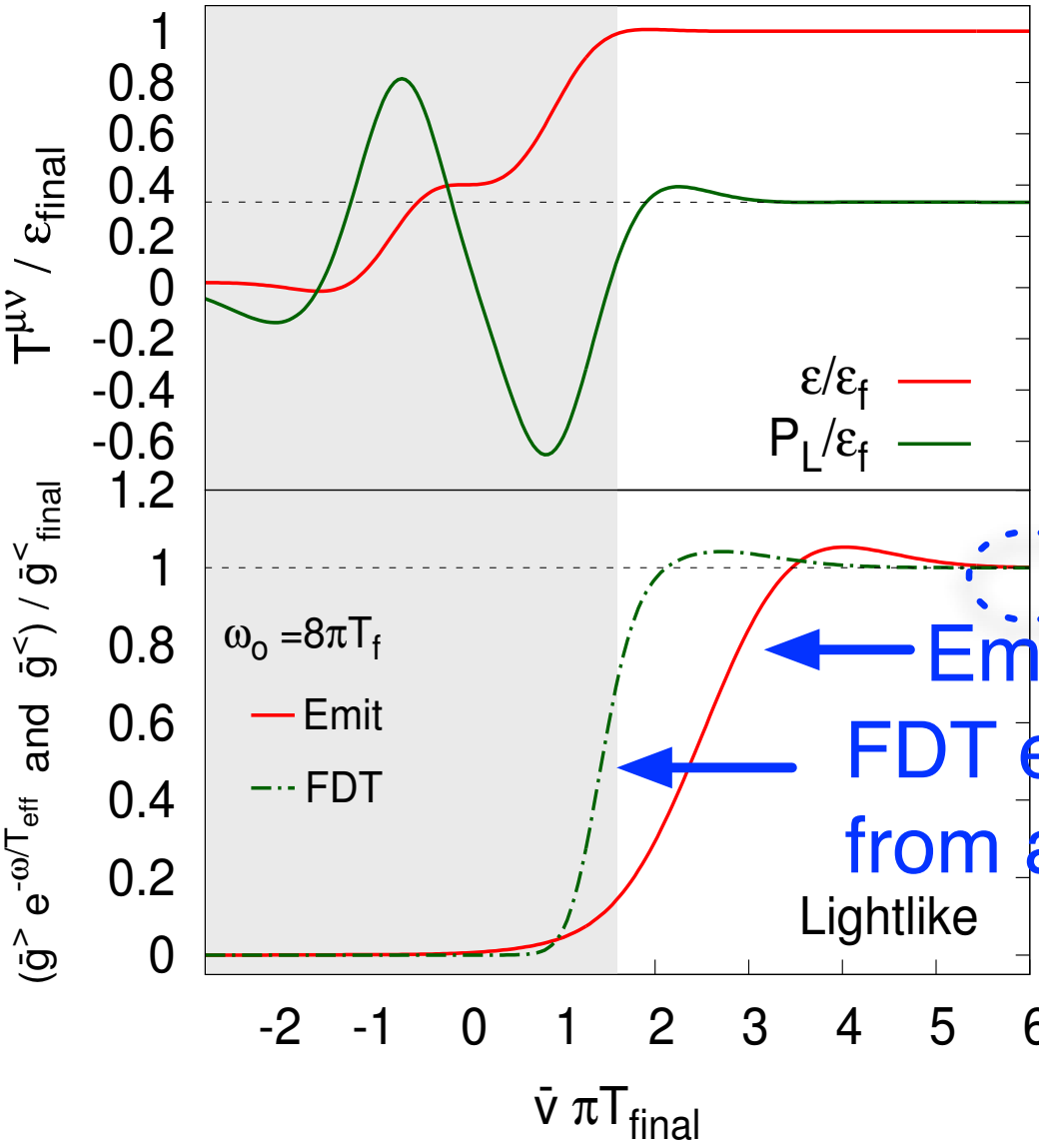


Lightlike emission rate $\omega_o = 8\pi T_f$



stress equilibrates

Lightlike emission rate $\omega_o = 8\pi T_f$

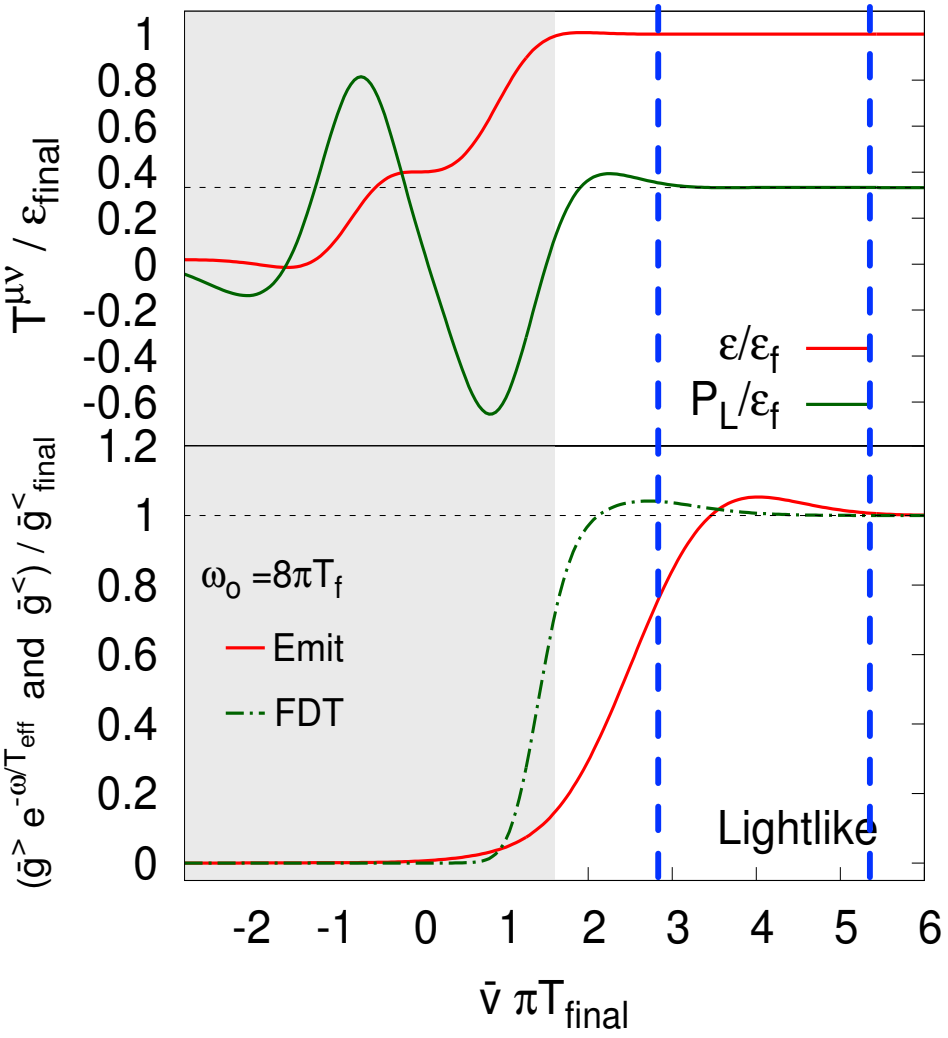


FDT satisfied

Emission rate

FDT estimate for emission from absorption

Lightlike emission rate $\omega_o = 8\pi T_f$



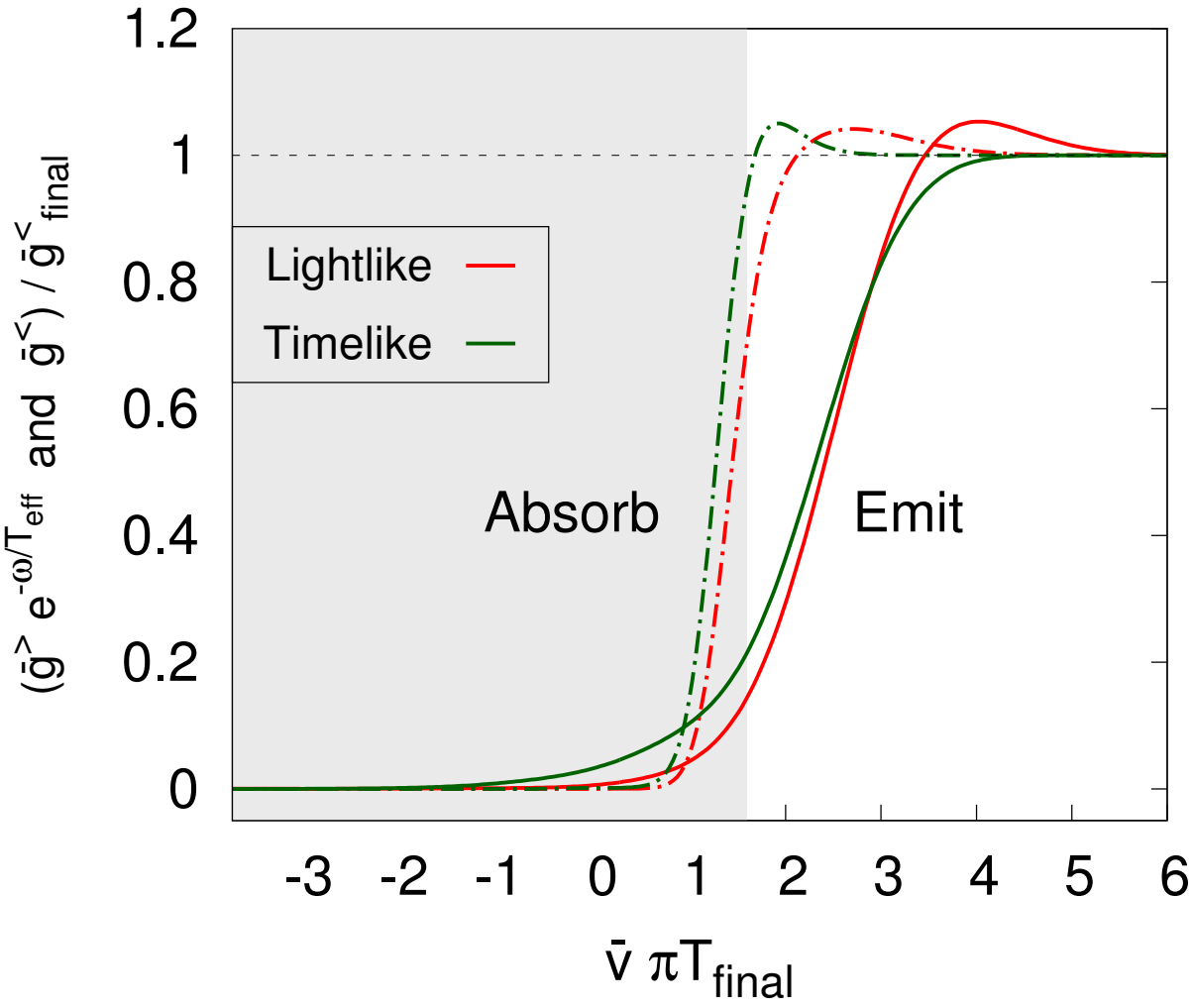
First the stress/geometry equilibrates

then

the emission rate equilibrates

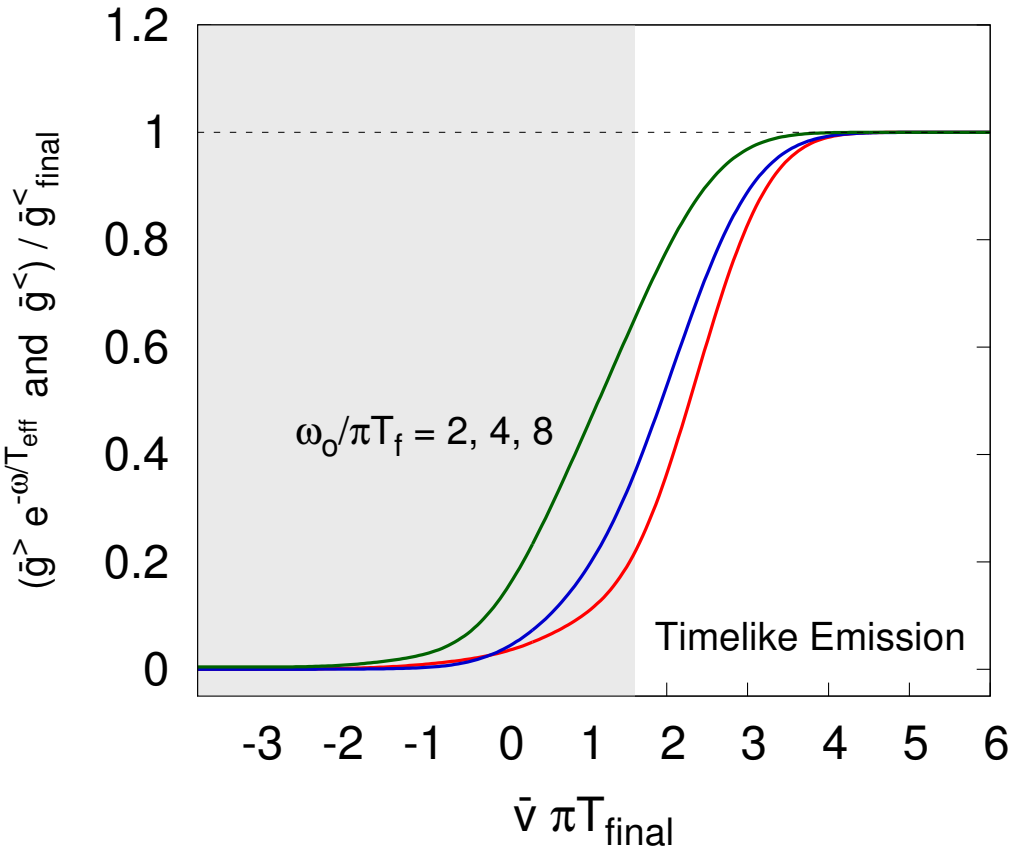
Takes a time $\tau_{\rm fly} \sim 2.5/\pi T$ to reach the boundary

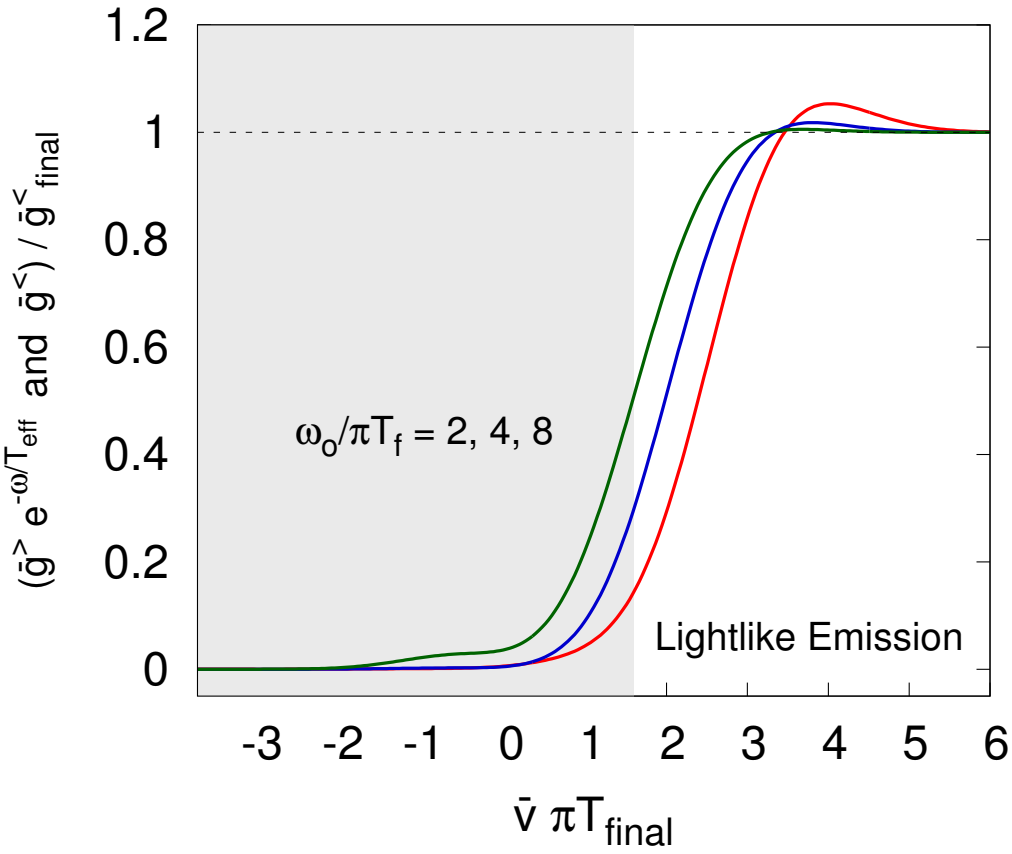
Lightlike emission versus timelike emission $\omega_o = 8\pi T_f$



Arguably – hard lightlike modes take longer to equilibrate

Frequency dependence of the emission rates





The harder the lightlike mode, the longer it takes to equilibrate. . . Expect

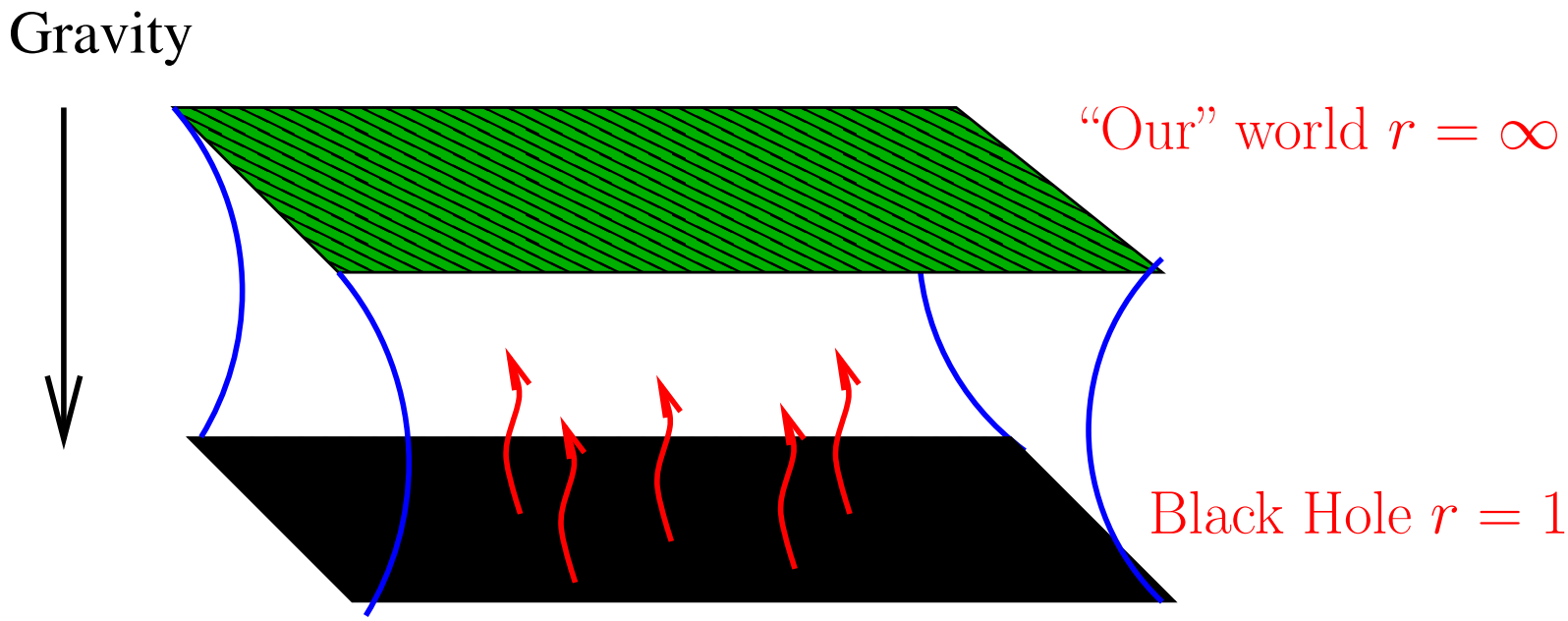
$$\tau_{\rm thermalize} \sim (\omega\sigma)^{1/4} \qquad \text{for} \qquad \omega \to \infty$$

but massive timkelike modes thermalize in a finite time. . . Expect

$$\tau_{\rm thermalize} \sim {\rm const} \qquad \omega \to \infty$$

Conclusions:

1. Picture



Gravity pulls down, but quantum fields fluctuate, reaching equilibrium

2. Picture gives a pattern to thermalization

   - Hard lightlike modes thermalize last
   - Thermalization times are controlled by the horizon coupling